

AN INVERSE PROBLEM FOR THE REFRACTIVE SURFACES WITH PARALLEL LIGHTING

ARAM L. KARAKHANYAN

ABSTRACT. In this article we examine the regularity of two types of weak solutions to a Monge-Ampère type equation which emerges in a problem of finding surfaces that refract parallel light rays emitted from source domain and striking a given target after refraction. Historically, ellipsoids and hyperboloids of revolution were the first surfaces to be considered in this context. The mathematical formulation commences with deriving the energy conservation equation for sufficiently smooth surfaces, regarded as graphs of functions to be sought, and then studying the existence and regularity of two classes of suitable weak solutions constructed from envelopes of hyperboloids or ellipsoids of revolution. Our main result in this article states that under suitable conditions on source and target domains and respective intensities these weak solutions are locally smooth.

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1. INTRODUCTION

Let $\mathcal{U} \subset \mathbb{R}^n$ be a bounded domain with smooth boundary and $u : \mathcal{U} \rightarrow \mathbb{R}$ a smooth function. By Γ_u we denote the graph of u . Let γ denote the unit normal of Γ_u . We think of Γ_u as a surface that dissects two distinct media.

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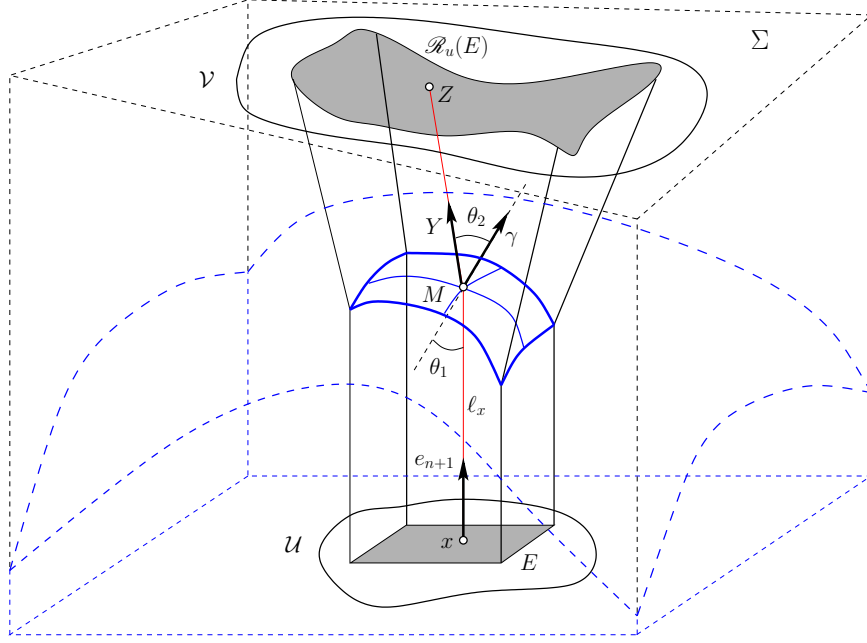


FIGURE 1. The blue dotted lines confine the boundary of media I.

From each $x \in \mathcal{U}$ we issue a ray ℓ_x parallel to e_{n+1} —the unit direction of the x_{n+1} axis in \mathbb{R}^{n+1} . Then ℓ_x strikes Γ_u , the surface separating the two media I and II, refracts into the second media II and strikes the receiver surface Σ , see Figure 1. Let Y be the unit direction of the refracted ray.

If γ is the unit normal at $M = (x, u(x)) \in \mathbb{R}^{n+1}$ where ℓ_x strikes Γ_u then from the refraction law we have

$$(1.1) \quad \frac{\sin \theta_1}{\sin \theta_2} = \frac{n_2}{n_1},$$

where n_1, n_2 are the refractive indices of the media I and II respectively, dissevered by the interface Γ_u , θ_1 and θ_2 are the angles between ℓ_x and γ , and between Y and γ , respectively, see Figure 1.

Suppose that the intensity of light on \mathcal{U} is $f \geq 0$ and let \mathcal{V} be the set of points where the refracted rays strike the receiver Σ . Denote by $g \geq 0$ the gain intensity on \mathcal{V} . For each $\mathcal{U}' \subset \mathcal{U}$ let \mathcal{V}' be the set of points where the rays, issued from \mathcal{U}' and refracted off Γ_u , strike Σ . Thus u generates the refractor mapping

$$Z_u : \mathcal{U} \longrightarrow \mathcal{V}$$

and the illuminated domain on Σ , corresponding to $\mathcal{U}' \subset \mathcal{U}$, is $\mathcal{V}' = Z_u(\mathcal{U}')$. If Γ_u is a perfect refractor, then one would have the energy balance equation (in local form)

$$(1.2) \quad \int_{\mathcal{U}'} f = \int_{\mathcal{V}' = Z_u(\mathcal{U}')} g.$$

The main problem that we are concerned with is formulated below:

Problem. Assume that we are given a smooth surface Σ in \mathbb{R}^{n+1} , a pair of bounded smooth domains $\mathcal{U} \subset \Pi = \{X \in \mathbb{R}^{n+1} : X^{n+1} = 0\}$ and $\mathcal{V} \subset \Sigma$ and a pair of nonnegative, integrable functions $f : \mathcal{U} \rightarrow \mathbb{R}$ and $g : \mathcal{V} \rightarrow \mathbb{R}$ such that the energy balance condition holds

$$(1.3) \quad \int_{\mathcal{U}} f = \int_{\mathcal{V}} g d\mathcal{H}^n.$$

Find a function $u : \mathcal{U} \rightarrow \mathbb{R}$ such that the following two conditions are fulfilled

$$(RP) \quad \begin{cases} \int_{\mathcal{U}'} f = \int_{Z_u(\mathcal{U}')} g, \text{ for any measurable } \mathcal{U}' \subset \mathcal{U} \\ Z_u(\mathcal{U}) = \mathcal{V}. \end{cases}$$

Problems of this kind appear in geometric optics [14] page 315. In the 17th century Descartes posed a similar problem with target set \mathcal{V} being a single point, say $\mathcal{V} = \{Z_0\}$. It was observed that the ellipsoids and hyperboloids of revolution with focal axis parallel to e_{n+1} will solve this problem if Z_0 is one of the foci. The case of general target \mathcal{V} can be treated via approximation argument, namely by constructing a solution from ellipsoids or hyperboloids for finite set $\mathcal{V} = \{Z_1, \dots, Z_m\}$ and then letting $m \rightarrow \infty$. Moreover, the eccentricity of these surfaces is fixed and determined by the refractive indices n_1 and n_2 . To see this we take advantage of some well-known facts from geometric optics and record them here for further reference, see [19]. Let $H(x) = Z^{n+1} - a\varepsilon - \frac{a}{b}\sqrt{b^2 + |x - x_0|^2}$ be the lower sheet of a hyperboloid of revolution with focal axis passing through the point $x_0 \in \mathcal{U}$ and parallel to e_{n+1} , see Section 8. Similarly, we define the lower half of an ellipsoid of revolution $E(x) = Z^{n+1} - a\varepsilon - \frac{a}{b}\sqrt{b^2 - |x - x_0|^2}$. If n_1 and n_2 are the refractive indices of media I and II respectively then

$$(1.4) \quad \varepsilon = \frac{n_1}{n_2} = \frac{\sin \theta_2}{\sin \theta_1} = \begin{cases} \frac{\sqrt{a^2 - b^2}}{a} < 1 & \text{for ellipsoids,} \\ \frac{\sqrt{a^2 + b^2}}{a} > 1 & \text{for hyperboloids.} \end{cases}$$

Here ε is the eccentricity, see [19]. Since ε is fixed we can drop the dependence of E and H from $b = a\sqrt{|\varepsilon^2 - 1|}$ and take

$$(1.5) \quad E(x, a, Z) = Z^{n+1} - a\varepsilon - a\sqrt{1 - \frac{(x - z)^2}{a^2(1 - \varepsilon^2)}}, \quad \text{if } \varepsilon < 1,$$

$$(1.6) \quad H(x, a, Z) = Z^{n+1} - a\varepsilon - a\sqrt{1 + \frac{(x - z)^2}{a^2(\varepsilon^2 - 1)}}, \quad \text{if } \varepsilon > 1.$$

We also define the constant

$$(1.7) \quad \kappa = \frac{\varepsilon^2 - 1}{\varepsilon^2}$$

which will prove to be useful, in a number of computation to follow.

2. MAIN THEOREMS

Let Σ be the receiver surface defined implicitly

$$(2.1) \quad \Sigma = \{Z \in \mathbb{R}^{n+1} : \psi(Z) = 0\}$$

where $\psi : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ is a smooth function. If $u \in C^2(\mathcal{U})$ then the first condition in **(RP)**, after using change of variables, results a Monge-Ampère type equation for u , whereas the second one plays the role of boundary condition for u . More precisely we have the following

Theorem A. *Let $u \in C^2(\mathcal{U})$ be a solution to **(RP)**. Then*

- 1° $Y = \varepsilon \left(\frac{\kappa Du}{1+q}, 1 - \frac{\kappa}{1+q} \right)$ is the unit direction of refracted ray,
- 2° u solves the equation

$$(2.2) \quad \left| \det \left[\frac{q+1}{t\varepsilon\kappa} \{ \text{Id} - \kappa\varepsilon^2 Du \otimes Du \} + D^2u \right] \right| = \left| -\varepsilon q \left[\frac{q+1}{t\varepsilon\kappa} \right]^n \frac{\nabla\psi \cdot Y}{|\nabla\psi|} \frac{f}{g} \right|,$$

where

$$(2.3) \quad q(x) = \sqrt{1 - \kappa(1 + |Du|^2)}, \quad \kappa = \frac{\varepsilon^2 - 1}{\varepsilon^2}$$

and t is the stretch function defined in (4.10) via an implicit relation $\psi(x + e_{n+1}u(x) + Yt) = 0$.

If the receiver Σ is a plane then taking $\psi(Z) = Z \cdot \xi + \xi_1$ we find that $t = -[Y \cdot \xi_0]^{-1}(x + u(x)e_{n+1} + \xi_1)$. In particular for the horizontal plane $X^{n+1} = m$, with some constant $m > 0$, one has

$$t = \frac{m - u}{Y^{n+1}} = (m - u) \frac{q + 1}{\varepsilon(1 - \kappa + q)}.$$

Quadric Σ is another example of receiver for which t can be computed explicitly. In general t is a function of $x, u(x)$ and $Du(x)$ which may not have simple explicit form. However, in terms of applications the case of planar receiver is of particular interest, since the flat screens are easy to construct. The method of the stretch function was introduced in [12, 13] to treat the near-field reflection problem. The equation for a near-field *refraction* problem with point source is derived in [6], [10].

Next, we need to introduce the notion of weak solution of (2.2). It will allow us to develop the existence theory along the lines of the classical Monge-Ampère equation. To this end, we say that $u : \mathcal{U} \rightarrow \mathbb{R}$ is upper (resp. lower) admissible with respect to \mathcal{V} if for any $x \in \mathcal{U}$ there is a hyperboloid $H(\cdot, a, Z)$ (resp. ellipsoid $E(\cdot, a, Z)$) with focus $Z \in \mathcal{V}$ such that $H(\cdot, a, Z)$ (resp. $E(\cdot, a, Z)$) touches u from above (resp. below) at x . Such $H(\cdot, a, Z)$ (resp. $E(\cdot, a, Z)$) is called supporting hyperboloid (resp. ellipsoid) of u at x . To fix the ideas we consider the class of upper admissible function and denote it by $\overline{\mathbb{W}}_H(\mathcal{U}, \mathcal{V})$. The class of lower admissible functions is denoted by $\underline{\mathbb{W}}_E(\mathcal{U}, \mathcal{V})$. For each $u \in \overline{\mathbb{W}}_H(\mathcal{U}, \mathcal{V})$ we define the mapping $\mathcal{S}_u : \mathcal{V} \rightarrow \mathcal{U}$ by

$$\mathcal{S}_u(Z) = \{x \in \mathcal{U} : \exists a > 0 \text{ such that } H(\cdot, a, Z) \text{ is a supporting hyperboloid of } u \text{ at } x\},$$

and take

$$\beta_{u,f}(E) = \int_{\mathcal{S}_u(E)} f(x) dx, \quad E \subset \mathcal{V}.$$

Furthermore, we also consider the mapping $\mathcal{R}_u : \mathcal{U} \rightarrow \mathcal{V}$ defined by

$$\mathcal{R}_u(x) = \{Z \in \mathcal{V} : \text{there is a supporting hyperboloid } H(\cdot, a, Z) \text{ of } u \text{ at } x\}$$

and associate the following set function

$$\alpha_{u,g}(E) = \int_{\mathcal{R}_u(E)} g d\mathcal{H}^n, \quad E \subset \mathcal{V}.$$

Notice that for smooth u , the mapping \mathcal{S}_u is the inverse of \mathcal{R}_u .

With the aid of these set functions $\alpha_{u,g}$ and $\beta_{u,f}$ we can introduce two notions of weak solution to **(RP)**, called A and B type weak solutions, respectively. It is not hard to see that $\beta_{u,f}$ is in fact σ -additive measure, while for $\alpha_{u,g}$ it is less obvious. Towards proving this the major obstruction is to show that \mathcal{R}_u is one-to-one modulo a set of vanishing \mathcal{H}^n measure on Σ . This is circumvented by introducing the Legendre-like transformation $v(z)$ of an admissible function $u(x)$ in Section 11 defined as an upper envelope of some function of $\text{dist}(Z, X)$ for $Z \in \mathcal{V}$ and $X \in \mathcal{U}$. In order to infer that $v(z)$ is semi-concave (which in turn will lead to σ -additivity of $\alpha_{u,g}$) we assume that (2.5) is fulfilled. That done, one can show that an A -type weak solution exists in the sense of Definition 11.2. Note that once we found the Legendre-like transformation then our problem can be treated as a prescribed Jacobian type equation discussed in [24]. However one still has to check all conditions formulated there in order to trigger the theory. Furthermore, the construction of locally smooth solution for **(RP)** is very complicated and require a careful analysis of Dirichlet's problem. This issues are addressed in Lemma 10.2 and Section 13.

If, for a moment, we take the existence of A -type weak solution for granted, the question about its regularity is even more complex. To set stage for the weak solutions we assume that $\Sigma = \{Z \in \mathbb{R}^{n+1} : \psi(Z) = 0\}$ and $\psi : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ being a smooth function. Clearly, some conditions must be imposed on ψ to guarantee, among other things, that the right hand side of the equation (2.2) is well defined, at least for smooth solutions.

To this end we enlist the following conditions to be used in the construction of weak solutions and proving their smoothness.

$$(2.4) \quad \nabla\psi(Z) \cdot (X - Z) > 0 \quad \forall X \in (\mathcal{U} \times [0, m_0]), \forall Z \in \Sigma \text{ and for some large constant } m_0 > 0,$$

$$(2.5) \quad \text{dist}(\mathcal{U}, \mathcal{V}) > 0,$$

$$(2.6) \quad \mathcal{V} \text{ is } R\text{-convex with respect to } \mathcal{U}, \text{ see Definition 10.2,}$$

$$(2.7) \quad f, g > 0,$$

$$(2.8) \quad \frac{1}{t} \left[\frac{t\varepsilon\kappa}{q+1} \right]^2 \text{II} + \frac{\kappa}{q} \frac{\psi_{n+1}}{|\nabla\psi|} \left(\text{Id} + \kappa \frac{p \otimes p}{q^2} \right) < 0, \quad \text{if } \kappa > 0,$$

where II is the second fundamental form of Σ and $p = Du(x)$. The subdomain of $\mathcal{U} \times [0, \infty)$ where (2.4)-(2.8) are simultaneously satisfied is called the *regularity domain* \mathcal{D} .

It is worthwhile to explain the meaning of these conditions: the first one (2.4) means that the reflected rays do not strike Σ tangentially, otherwise Σ would not detect the gain intensity at the tangential points, i.e. at the points where $\nabla\psi(Z) \cdot (X - Z) = 0$. On the technical level, however, it allows to apply the inverse function theorem to recover the stretch function $t = t(x, u, Du)$. It is worth pointing out that (2.4) holds for a large class of surfaces Σ . To see this it is enough to notice that there is a positive constant $c(\varepsilon)$, depending only on ε such that $Y^{n+1} \in [c(\varepsilon), 1]$. In other words the unit directions Y of refracted rays remain within the cone $c(\varepsilon) \leq Y^{n+1} \leq 1$. Indeed, if u is differentiable at x then $Y^{n+1}(x) = \varepsilon[1 + (\sqrt{\cos^2 \theta_1 - \kappa} - \cos \theta_1)\gamma^{n+1}]$ from refraction law, see (4.4) and Figure 1. Here $\kappa = 1 - \varepsilon^{-2}$. If u is not differentiable at x , we interpret γ as one of the normals of supporting planes of admissible u at $x \in \mathcal{U}$ since u is concave (resp. convex) if u is upper (resp. lower) admissible, see Section 8. Thus if $p \in \partial u(x)$, where ∂u is the subdifferential of u at x , then $\gamma = \left(\frac{p}{\sqrt{1+|p|^2}}, \frac{1}{\sqrt{1+|p|^2}} \right)$ and $\cos \theta_1 = \frac{1}{\sqrt{1+|p|^2}}$. Consequently if u is lower admissible then $Y^{n+1} \geq \varepsilon$ if $\kappa < 0$, i.e. $\varepsilon < 1$ and hence $c(\varepsilon) = \varepsilon < 1$. On the other hand if $\kappa > 0$ then for any $u \in \mathbb{H}(\mathcal{U}, \mathcal{V})$ we have

$$(2.9) \quad \sup_{\mathcal{U}} |Du| < \frac{1}{\sqrt{\varepsilon^2 - 1}} \quad \text{if } \kappa = \frac{\varepsilon^2 - 1}{\varepsilon^2} > 0.$$

This simply follows from the fact that supporting hyperboloids control the magnitude of the gradient of u , see Lemma 8.1. But in its turn $|DH|$, for any hyperboloid H given by (1.6), satisfies the estimate (2.9). Because $\gamma^{n+1} = \cos \theta_1 = \frac{1}{\sqrt{1+|p|^2}}$ (see Figure 1 and the derivation of (4.7)) and $|p| < \frac{1}{\sqrt{\varepsilon^2 - 1}}$ we infer that

$$(\sqrt{\cos^2 \theta_1 - \kappa} - \cos \theta_1)\gamma^{n+1} = \frac{-\kappa}{\sqrt{\cos^2 \theta_1 - \kappa} + \cos \theta_1} \gamma^{n+1} > -\kappa$$

and consequently $Y^{n+1} > \varepsilon(1 - \kappa) = \varepsilon^{-1} < 1$. Thus for $\varepsilon > 1$ we can take $c(\varepsilon) = \varepsilon^{-1}$. From here we see that (2.4) holds for any horizontal receiver $Z^{n+1} = m$, for large $m > 0$. More generally if Σ is concave in Z^{n+1} direction and the normal mapping of Σ is strictly inside of the cone $c(\varepsilon) < Y^{n+1}$ on the unit sphere then (2.4) holds true. This leads to the following cone condition for the unit directions of refracted rays

$$(2.10) \quad 0 < c(\varepsilon) \leq Y^{n+1} \leq 1.$$

The second condition (2.5) assures that the Legendre-like transformation $v(z)$ for an admissible function u is well defined as an envelope of C^1 smooth functions, in particular $v(z)$ is semi-concave and hence differentiable almost everywhere, see Section 11. This yields that $\alpha_{u,g}$ is a Radon measure.

The next two conditions (2.6) and (2.7) assure that B -type solution is also of A -type and therefore one gets the existence of A -type weak solutions in some indirect way using the methods of [4], [26]. That done, we can approximate \mathcal{V} by R -convex domains and show the existence of A -type weak solutions without assuming (2.6), see Theorem C4.

Last condition (2.8), which is crucial for regularity of weak solutions, deserves special attention because it is the most sophisticated one. In fact the next theorem is entirely devoted to the verification of (2.8).

Theorem B. *Let u be a $C^2(\mathcal{U})$ solution of (2.2) and Π be the second fundamental form of $\Sigma = \{Z \in \mathbb{R}^{n+1} : Z^{n+1} = \varphi(z)\}$. Let $\kappa > 0, \Pi < 0$ then there is a constant $\lambda > 0$ such that if $\text{dist}(\mathcal{U}, \mathcal{V}) > \lambda$ then (2.8). Furthermore, for $\kappa < 0, \Pi > 0$ there is a constant $\hat{\lambda} > 0$ such that if $\text{dist}(\mathcal{U}, \mathcal{V}) > \hat{\lambda}$ then the opposite inequality in (2.8) holds for all $x \in \mathcal{U}$.*

If Σ is a graph, say $Z^{n+1} = \varphi(z)$ then (2.8) can be rewritten as

$$\frac{1}{t} \left[\frac{t\varepsilon\kappa}{q+1} \right]^2 D^2\varphi + \frac{\kappa}{q} \left(\text{Id} + \kappa \frac{p \otimes p}{q^2} \right) < 0, \quad \text{if } \kappa > 0.$$

In lieu of (2.10) this assumption on Σ is not restrictive. In addition, Theorem B suggests that it is convenient to think of Σ as an unbounded convex (resp. concave) surface without boundary if $\kappa < 0$ (resp. $\kappa > 0$) by extending φ to \mathbb{R}^n as a convex function $\tilde{\varphi}$ such that $\varphi(z) \rightarrow \pm\infty$ as $|z| \rightarrow \infty$. We will take advantage of such extension of φ (and hence Σ) in Section 8.4 and Lemma 10.2, see also Remark 7.1.

Now we are ready to formulate our main existence result.

Theorem C. *1 If $f, g \geq 0$ and (1.3) and (2.4) hold then there is a B -type weak solution provided that the condition below*

$$(2.11) \quad Z^{n+1} \geq \left[\frac{2}{\varepsilon-1} + \frac{1}{\sqrt{\varepsilon^2-1}} \right] \rho(z)$$

is satisfied. Here $\rho(z) = \inf\{R > 0 : \mathcal{U} \subset B_R(z)\}$ is the maximal visibility radius from $z \stackrel{\text{def}}{=} \hat{Z} \in \hat{\mathcal{V}}$,

- 2 if (2.4) and (2.5) hold then $\alpha_{u,g}$ is countably additive,*
- 3 if (2.4)-(2.6), (2.11) hold and $f \geq 0$ while $g > 0$ then B -type weak solution is also of A type,*
- 4 if we remove the R -convexity assumption but require the positivity of densities (2.7) and (2.4)-(2.5), (2.11) then again any B -type weak solution is also of A -type.*

The proof of Theorem C1 is by polyhedral approximation and utilising the confocal expansion of hyperboloids as described in Section 8.4. In this regard the condition (2.11) in Theorem C1 says that one can construct a B -type weak solution if there is sufficient span between Π and Σ . The existence of B -type weak solutions, constructed from an envelope of ellipsoids of revolution can be found in [7].

Our last result concerns with the smoothness of A -type weak solutions. We use the well-known method of comparing the mollified weak solution with that of Dirichlet's problem to the slightly modified equation in a small ball B . To this end one first has to obtain $C^{2,\alpha}$, $\alpha \in (0, 1]$ estimates in \overline{B} for the solutions of mollified equations and after that making sure that uniform C^2 estimates hold in, say, $\frac{1}{2}B$. Then passing to limit and using the comparison principle the result will follow. The construction of weak solutions to Dirichlet's problem is based on Perron's method and follows the approach developed by Xu-Jia Wang in [27] where a far field reflector design

problem is studied. Our research is inspired by [27] and subsequent developments in [12], [13] [11]. For more recent results on this problem see [15]. The global C^2 estimates for the solution of Dirichlet's problem for the regularised equation follow from [9] whereas the local uniform estimates in $\frac{1}{2}B$ are established in [18], see also [16] for the global regularity of near field reflector problem with point source of a light. Thus we have the following theorem

Theorem D. *Let f, g be C^2 smooth functions such that $\lambda \leq f, g \leq \Lambda$ for some constants $\Lambda > \lambda > 0$ and the conditions (2.4)- (2.8) are satisfied. Then A -type weak solutions of (\mathbf{RP}) are locally C^2 regular in \mathcal{U} .*

The conditions (2.4)- (2.8) cannot be relaxed as one may easily construct counterexamples to regularity in the spirit of those in [12], [13]. For instance let us examine (2.6) (see also Remark 12.3), if we take a two point target $\mathcal{V} = \{Z_1\} \cup \{Z_2\}$ and consider $H(x) = \min[H(x, a_1, Z_1), H(x, a_2, Z_2)]$ such that these hyperboloids $H(\cdot, a_i, Z_i), i = 1, 2$ have non empty intersection over \mathcal{U} . Then approximating \mathcal{V} by smooth R-convex sets \mathcal{V}_t we obtain a sequence of admissible functions H_t , solving the refractor problem with target \mathcal{V}_t , and converging to H as $t \rightarrow 0$. But if t is sufficiently close to 0 then H_t cannot remain C^1 smooth because otherwise the limit H would also be C^1 which is impossible., see [12] for more discussion on such constructions. We would also like to point out a recent paper of Guti rred and Tournier [8] where the authors study the local $C^{1,\alpha}$ regularity of reflector/refractor problems without using the explicit form of the equation. There one can find a detailed account of the case when the supporting functions are ellipsoids of revolution. Our work contributes in this direction only by deriving the explicit equation for general receiver Σ and establishing a simple form of the corresponding regularity condition (involving the second fundamental form of Σ) for the existence of C^2 smooth solutions, see Section 7.3. In this paper, however, we mainly focus on the case when the supporting functions are the hyperboloids of revolution.

The rest of the paper is organized as follows: in the next section we derive the main formulae. Then we prove Theorem A in Section 4. The main result there is Proposition 5.1 from which the proof of Theorem A easily follows. Section 5 contains some preliminary discussion on the condition (2.8) and after that in Section 6 we give the proof of Theorem B. The admissible functions are introduced in Section 7 where we also exhibit some interesting properties of hyperboloids of revolution, notably the dual admissibility and confocal expansion. Employing the polyhedral approximation technique and weak convergence of measures $\beta_{u,f}$ we prove Theorem C1 in Section 8. The first direct application of (2.8) is given in Lemma 10.1, which is G. Loeper's geometric interpretation of the A3 condition from [18]. A direct consequence of this is Lemma 10.2 stating that a suitable dilation of an admissible function by a paraboloid of revolution can be approximated via smooth subsolutions of (2.2). This is a crucial ingredient in the proof of Theorem D. Next we introduce the Legendre-like transformation of an admissible u and conclude Theorem C2. The proofs of Theorem C3-4 follow from a comparison of A and B type weak solutions by extending the results of Luis Caffarelli [4] and John Urbas [26] for the classical Monge-Amp re equation to (2.2). This is done in Section 11. The last two sections are devoted to the study of the higher regularity of A -type weak solutions. We follow the classical approach developed by A. Pogorelov for the classical Monge-Amp re equation, see [20], [21]. Therefore, we first prove the solvability of weak Dirichlet's problem when the boundary data is given as the trace of an A -type weak subsolution. That done, the uniqueness follows from comparison principle stated in Proposition 13.1. Finally in Section 13 we give the proof of our main regularity result, Theorem D.

3. NOTATIONS

C, C_0, C_n, \dots	generic constants,
Π	$\Pi = \mathbb{R}^n \times \{0\}$,

$\overline{\Omega}$	closure of a set Ω ,
$\partial\Omega$	boundary of a set Ω ,
$\widehat{\Omega}$	the projection of $\Omega \subset \mathbb{R}^{n+1}$ on Π ,
\widehat{X}	$(x_1, x_2, \dots, x_n, 0)$ projection of $X = (x_1, x_2, \dots, x_n, x_{n+1})$,
ε	eccentricity,
κ	$\kappa = \frac{\varepsilon^2 - 1}{\varepsilon^2}$,
\mathcal{H}^n	n -dimensional Hausdorff measure on Σ ,
∂_i	partial derivate with respect to x_i variable,
Du	the gradient of a function u ,
$\rho(z)$	$\inf\{R > 0 : \mathcal{U} \subset B_z(R)\}$ is the maximal visibility radius from $z \stackrel{\text{def}}{=} \widehat{Z} \in \widehat{\mathcal{V}}$,
q	see (2.3),
$\mathbb{H}(\mathcal{U}, \Sigma)$	the class of hyperboloids of revolution with focal axis parallel to Z^{n+1} and upper focus on Σ ,
$\mathbb{H}_{a_0}^+(\mathcal{U}, \mathcal{V})$	hyperboloids from $\mathbb{H}(\mathcal{U}, \mathcal{V})$ which are nonnegative in \mathcal{U} and $a > a_0$ for some fixed a_0 ,
$\overline{\mathbb{W}}_{\text{H}}, \underline{\mathbb{W}}_{\text{E}}$	upper and lower admissible functions, see Lemma 8.1,
$\overline{\mathbb{W}}_{\text{H}}^0(\mathcal{U}, \mathcal{V})$	polyhedral admissible functions.

4. MAIN FORMULAE

In this section we derive the Monge-Ampère type equation (2.2) manifesting the energy balance condition (1.2) in the refractor problem **(RP)**, see Introduction.

4.1. Computing Y . We first compute the unit direction of the refracted ray. Denote by γ the unit normal to the graph of u , that is

$$(4.1) \quad \gamma = \frac{(-D_1 u, \dots, -D_n u, 1)}{\sqrt{1 + |Du|^2}}.$$

Since ℓ_x, Y and γ lie in the same hyperplane we have

$$(4.2) \quad Y = \mathcal{A}e_{n+1} + \mathcal{B}\gamma,$$

for some coefficients \mathcal{A} and \mathcal{B} . Computing the scalar products $Y \cdot \gamma$ and $Y \cdot e_{n+1}$ we obtain the following equations (cf. (1.1))

$$\begin{cases} \cos \theta_2 = \mathcal{A} \cos \theta_1 + \mathcal{B}, \\ \cos(\theta_1 - \theta_2) = \mathcal{A} + \mathcal{B} \cos \theta_1. \end{cases}$$

Multiplying the first equation by $\cos \theta_1$ and subtracting from the second one we conclude

$$\mathcal{A} = \frac{\sin \theta_2}{\sin \theta_1}, \quad \mathcal{B} = \cos \theta_2 - \mathcal{A} \cos \theta_1.$$

Recalling our notations

$$(4.3) \quad \kappa = \frac{\varepsilon^2 - 1}{\varepsilon^2}, \quad \varepsilon = \frac{n_1}{n_2},$$

we see that $\mathcal{A} = \varepsilon$. Furthermore

$$n_2^2 - n_2^2 \cos^2 \theta_2 = n_2^2 \sin^2 \theta_2 = n_1^2 \sin^2 \theta_1 = n_1^2 - n_1^2 \cos^2 \theta_1.$$

Dividing both sides of this identity by n_2^2 we obtain

$$\cos^2 \theta_2 = \varepsilon^2 \cos^2 \theta_1 - (\varepsilon^2 - 1) = \varepsilon^2 (\cos^2 \theta_1 - \kappa).$$

Therefore from $\mathcal{A} = \varepsilon$ we conclude that $\mathcal{B} = \varepsilon(\sqrt{\cos^2 \theta_1 - \kappa} - \cos \theta_1)$. Returning to (4.2) we infer that the unit direction of the refracted ray is

$$(4.4) \quad Y = \varepsilon \left(e_{n+1} + (\sqrt{\cos^2 \theta_1 - \kappa} - \cos \theta_1) \gamma \right).$$

Notice that (4.1) implies

$$\cos \theta_1 = \gamma \cdot e_{n+1} = \frac{1}{\sqrt{1 + |Du|^2}}.$$

Consequently, denoting $Y = (Y^1, Y^2, \dots, Y^n, Y^{n+1})$ and $y \in \mathbb{R}^n$, the projection of Y onto $\Pi = \{X \in \mathbb{R}^{n+1} : X^{n+1} = 0\}$, (i.e. $y = (Y^1, Y^2, \dots, Y^n, 0)$) we get

$$(4.5) \quad \begin{aligned} y &= -\varepsilon \frac{Du}{\sqrt{1 + |Du|^2}} \left(\sqrt{\cos^2 \theta_1 - \kappa} - \cos \theta_1 \right) \\ &= \frac{\varepsilon \kappa Du}{\sqrt{1 + |Du|^2}} \frac{1}{\sqrt{\cos^2 \theta_1 - \kappa} + \cos \theta_1} \\ &= \varepsilon \kappa \frac{Du}{\sqrt{1 - \kappa(1 + |Du|^2)} + 1}. \end{aligned}$$

From this computation it follows that

$$(4.6) \quad Y^{n+1} = \varepsilon \left(1 - \frac{\kappa}{1 + \sqrt{1 - \kappa(1 + |Du|^2)}} \right).$$

Combining (4.5) and (4.6) we obtain

$$(4.7) \quad Y = \varepsilon \left(\frac{\kappa Du}{1 + \sqrt{1 - \kappa(1 + |Du|^2)}}, 1 - \frac{\kappa}{1 + \sqrt{1 - \kappa(1 + |Du|^2)}} \right).$$

If we use the notation $q(x) = \sqrt{1 - \kappa(1 + |Du|^2)}$ (see (2.3)) then (4.7) takes the form

$$(4.8) \quad Y = \varepsilon \left(\frac{\kappa Du}{1 + q}, 1 - \frac{\kappa}{1 + q} \right).$$

Notice that by (4.7) $Y^{n+1} > 0$ for all values of κ .

4.2. Stretch function. Assume that ψ is a smooth function $\psi : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$, and the receiver Σ is given as the zero set of ψ

$$(4.9) \quad \Sigma = \{Z \in \mathbb{R}^{n+1} : \psi(Z) = 0\}.$$

Let us represent the mapping $Z : \mathcal{U} \rightarrow \Sigma$ in the following form

$$(4.10) \quad Z = x + e_{n+1}u(x) + Yt,$$

where $t = t(x, u(x), Du(x))$ is determined from the equation $\psi(Z) = 0$ and is called the *stretch function*. It is worthwhile to point out that the stretch function t can be explicitly computed for a wide class of elementary surfaces. For instance, if Σ is the horizontal plane $Z^{n+1} = m > 0$ then from simple geometric considerations one finds that

$$t = \frac{m - u}{Y^{n+1}}$$

where Y^{n+1} is given by (4.6).

In lemma to follow we denote by z the projection of Z onto Π , that is $z = x + ty$.

Lemma 4.1. *Let $dS_{\mathcal{U}}$ and $dS_{\mathcal{V}}$ be the area elements on \mathcal{U} and $Z(\mathcal{U}) = \mathcal{V} \subset \Sigma$ respectively and z being the projection of Z onto $\Pi = \{Z \in \mathbb{R}^{n+1} : Z^{n+1} = 0\}$. Then we have*

$$(4.11) \quad J = \frac{dS_{\mathcal{V}}}{dS_{\mathcal{U}}} = \begin{vmatrix} Z_1^1 & \cdots & Z_n^1 & \nu^1 \\ \vdots & \ddots & \vdots & \vdots \\ Z_1^n & \cdots & Z_n^n & \nu^n \\ Z_1^{n+1} & \cdots & Z_n^{n+1} & \nu^{n+1} \end{vmatrix} \\ = -\frac{|\nabla\psi|}{\psi_{n+1}} \det Dz,$$

where ν is the unit normal of Σ .

Proof. The first equality in (4.11) follows from the change of variables formula. Differentiating the equality $\psi(Z) = 0$ by x_i we have that

$$\partial_i Z^{n+1} = -\frac{1}{\partial_{n+1}\psi} \sum_{k=1}^n \partial_i z^k \partial_{z_k} \psi.$$

Using this identity we multiply j -th row of matrix in (4.11) by $\partial_{z_j} \psi$ and subtract it from the $(n+1)$ st row in order to get

$$\det \begin{vmatrix} Z_1^1 & \cdots & Z_n^1 & \nu_1 \\ \vdots & \ddots & \vdots & \vdots \\ Z_1^n & \cdots & Z_n^n & \nu_n \\ Z_1^{n+1} & \cdots & Z_n^{n+1} & \nu_{n+1} \end{vmatrix} = -\frac{1}{\psi_{n+1}} \det \begin{vmatrix} Z_1^1 & \cdots & Z_n^1 & \nu_1 \\ \vdots & \ddots & \vdots & \vdots \\ Z_1^n & \cdots & Z_n^n & \nu_n \\ \sum_{k=1}^n \partial_1 z^k \partial_{z_k} \psi, \cdots, \sum_{k=1}^n \partial_n z^k \partial_{z_k} \psi, -\psi_{n+1} \nu_{n+1} \end{vmatrix} \\ = -\frac{1}{\psi_{n+1}} \det \begin{vmatrix} Z_1^1 & \cdots & Z_n^1 & \nu_1 \\ \vdots & \ddots & \vdots & \vdots \\ Z_1^n & \cdots & Z_n^n & \nu_n \\ 0, \cdots, 0, -\sum_{k=1}^{n+1} \psi_k \nu_k \end{vmatrix}.$$

Finally noting that $\nu = \frac{\nabla\psi}{|\nabla\psi|}$ the desired identity follows. \square

Lemma 4.2. *Let $C \in \mathbb{R}$ and $\xi, \eta \in \mathbb{R}^n$. Consider the matrix $\mu = \text{Id} + C\xi \otimes \eta = \delta_{ij} + C\xi^i \eta^j$ where $\text{Id} = \delta_{ij}$ is the identity matrix. Then the inverse matrix of μ is*

$$\det \mu = 1 + C\xi \cdot \eta, \\ \mu^{-1} = \text{Id} - \frac{C\xi \otimes \eta}{1 + C(\xi \cdot \eta)}.$$

Here and henceforth Id is the identity matrix.

Proof. Without loss of generality we assume that $\xi = e_1$ then $\det \mu = 1 + C\eta^1$. As for the second identity, it is a partucal case of Sherman-Morrison formula. \square

Finally, we derive a formula for the first order derivatives of the stretch function t . Let us differentiate the equation $\psi(Z) = 0$ with respect to x_j to get

$$\sum_{k=1}^n \psi_k (\delta_{kj} + t_j y^k + t y_j^k) + \psi_{n+1} (u_j + t_j Y^{n+1} + t Y_j^{n+1}) = 0.$$

From here we find

$$(4.12) \quad t_j = -\frac{1}{\nabla\psi \cdot Y} [\psi_j + \psi_{n+1}u_j + t(\nabla\psi \cdot Y_j)].$$

5. PROOF OF THEOREM A

In this section we prove Theorem A. We begin with a computation for the matrix Dz , where z is the projection of Z on to Π .

Proposition 5.1. *Let $u \in C^2(\mathcal{U})$ and Z be the corresponding refractor map, then with the same notations as in Lemma 4.1 we have*

$$(5.1) \quad Dz = \mu_1\mu_2 \left[\text{Id} - \kappa\varepsilon^2 Du \otimes Du + \frac{t\kappa\varepsilon}{1+h} D^2u \right],$$

where

$$(5.2) \quad \mu_1 = \text{Id} - \frac{y \otimes (\widehat{\nabla}\psi - y \frac{\psi_{n+1}}{Y^{n+1}})}{\nabla\psi \cdot Y}, \quad \mu_2 = \text{Id} + \kappa \frac{Du \otimes Du}{q(q+1)},$$

$q = \sqrt{1 - \kappa(1 + |Du|^2)}$ and

$$(5.3) \quad \widehat{\nabla}\psi = (\psi_1, \dots, \psi_n, 0).$$

In order to prove Proposition 5.1 we will need the following

Lemma 5.1. *Let $z(x), x \in \mathcal{U}$ be the projection of the mapping $Z(x)$ onto $\Pi = \{X \in \mathbb{R}^{n+1} : X^{n+1} = 0\}$. Then*

$$(5.4) \quad Dz = \mu_1 (\text{Id} - y \otimes [y + DuY^{n+1}] + tDy)$$

where μ_1 is defined by (5.2).

Proof. Introduce the matrix

$$(5.5) \quad \mu_0 = \delta_{ij} - y^i \frac{\psi_j + u_j \psi_{n+1}}{\nabla\psi \cdot Y}.$$

Using (4.12) and recalling $z = x + ty$ we compute

$$(5.6) \quad \begin{aligned} z_j^i &= \delta_{ij} + t_j y^i + t y_j^i \\ &= \delta_{ij} + t y_j^i - y^i \frac{1}{\nabla\psi \cdot Y} [\psi_j + \psi_{n+1}u_j + t(\nabla\psi \cdot Y_j)] \\ &= \delta_{ij} - y^i \underbrace{\frac{[\psi_j + u_j \psi_{n+1}]}{\nabla\psi \cdot Y}}_{\mu_0} + t \left[y_j^i - \frac{y^i (\nabla\psi \cdot Y_j)}{\nabla\psi \cdot Y} \right] \\ &= \mu_0 + t \left[y_j^i - \frac{y^i (\nabla\psi \cdot Y_j)}{\nabla\psi \cdot Y} \right]. \end{aligned}$$

In order to deal with the remaining matrix we recall that $(Y^{n+1})^2 = 1 - |y|^2$ and hence $Y_j^{n+1} = -\frac{yy_j}{Y^{n+1}}$. Consequently, setting $\widehat{\nabla}\psi = (\psi_1, \dots, \psi_n, 0)$ (see (5.3)) we infer

$$(5.7) \quad \begin{aligned} y_j^i - \frac{y^i (\nabla\psi \cdot Y_j)}{\nabla\psi \cdot Y} &= y_j^i - \frac{y^i}{\nabla\psi \cdot Y} \left(\widehat{\nabla}\psi \cdot y_j - \psi_{n+1} \frac{y \cdot y_j}{Y^{n+1}} \right) \\ &= y_j^i - \frac{y^i}{\nabla\psi \cdot Y} \left[\left(\widehat{\nabla}\psi - \frac{\psi_{n+1}}{Y^{n+1}} y \right) \cdot y_j \right]. \end{aligned}$$

Combining (5.6) and (5.7) we obtain the following formula for Dz , written in intrinsic form

$$\begin{aligned}
 (5.8) \quad Dz &= \mu_0 + t \left[\text{Id} - \frac{y \otimes \left(\widehat{\nabla} \psi - \frac{\psi_{n+1}}{Y^{n+1}} y \right)}{\nabla \psi \cdot Y} \right] Dy \\
 &= \mu_0 + t \mu_1 Dy \\
 &= \mu_1 (\mu_1^{-1} \mu_0 + t Dy)
 \end{aligned}$$

where the second equality follows from the definition of matrix μ_1 , see (5.2).

Next, we compute μ_1^{-1} . From Lemma 4.2 and the identity $[Y^{n+1}]^2 = 1 - |y|^2$ we get

$$\begin{aligned}
 (5.9) \quad \mu_1^{-1} &= \text{Id} + y \otimes \frac{\widehat{\nabla} \psi - \frac{\psi_{n+1}}{Y^{n+1}} y}{\nabla \psi \cdot Y - \left(\widehat{\nabla} \psi \cdot y - |y|^2 \frac{\psi_{n+1}}{Y^{n+1}} \right)} \\
 &= \text{Id} + \frac{Y^{n+1}}{\psi_{n+1}} y \otimes \left[\widehat{\nabla} \psi - \frac{\psi_{n+1}}{Y^{n+1}} y \right],
 \end{aligned}$$

where the last equality follows from the observation

$$\begin{aligned}
 (5.10) \quad \nabla \psi \cdot Y - \left(\widehat{\nabla} \psi \cdot y - |y|^2 \frac{\psi_{n+1}}{Y^{n+1}} \right) &= \psi_{n+1} Y^{n+1} + (1 - (Y^{n+1})^2) \frac{\psi_{n+1}}{Y^{n+1}} \\
 &= \frac{\psi_{n+1}}{Y^{n+1}}.
 \end{aligned}$$

It is convenient to rewrite this identity in the following form

$$(5.11) \quad \left[\widehat{\nabla} \psi \cdot y - |y|^2 \frac{\psi_{n+1}}{Y^{n+1}} \right] \frac{Y^{n+1}}{\psi_{n+1}} \frac{1}{\nabla \psi \cdot Y} = \frac{Y^{n+1}}{\psi_{n+1}} - \frac{1}{\nabla \psi \cdot Y}.$$

Consequently, we obtain

$$\begin{aligned}
 \mu_1^{-1} \mu_0 &= \left(\text{Id} + \frac{Y^{n+1}}{\psi_{n+1}} y \otimes \left[\widehat{\nabla} \psi - \frac{\psi_{n+1}}{Y^{n+1}} y \right] \right) \left(\text{Id} - y \otimes \frac{\widehat{\nabla} \psi + Du\psi_{n+1}}{\nabla \psi \cdot Y} \right) \\
 &= \text{Id} - y \otimes \frac{\widehat{\nabla} \psi + Du\psi_{n+1}}{\nabla \psi \cdot Y} + \frac{Y^{n+1}}{\psi_{n+1}} y \otimes \left[\widehat{\nabla} \psi - \frac{\psi_{n+1}}{Y^{n+1}} y \right] - \\
 &\quad - \left[\widehat{\nabla} \psi \cdot y - |y|^2 \frac{\psi_{n+1}}{Y^{n+1}} \right] \frac{Y^{n+1}}{\psi_{n+1}} \frac{1}{\nabla \psi \cdot Y} \left\{ y \otimes \widehat{\nabla} \psi + Du\psi_{n+1} \right\}.
 \end{aligned}$$

Applying (5.11) to the last term in this computation we get

$$\begin{aligned}
 \mu_1^{-1} \mu_0 &= \text{Id} - y \otimes \frac{\widehat{\nabla} \psi + Du\psi_{n+1}}{\nabla \psi \cdot Y} + \frac{Y^{n+1}}{\psi_{n+1}} y \otimes \left[\widehat{\nabla} \psi - \frac{\psi_{n+1}}{Y^{n+1}} y \right] - \\
 &\quad - \left[\frac{Y^{n+1}}{\psi_{n+1}} - \frac{1}{\nabla \psi \cdot Y} \right] \left\{ y \otimes \widehat{\nabla} \psi + Du\psi_{n+1} \right\} \\
 &= \text{Id} + \frac{Y^{n+1}}{\psi_{n+1}} y \otimes \left[\widehat{\nabla} \psi - \frac{\psi_{n+1}}{Y^{n+1}} y \right] - \\
 &\quad - \frac{Y^{n+1}}{\psi_{n+1}} \left\{ y \otimes \widehat{\nabla} \psi + Du\psi_{n+1} \right\} \\
 &= \text{Id} - y \otimes [y + Du\psi_{n+1}].
 \end{aligned}$$

Plugging in the computed form of $\mu_1^{-1} \mu_0$ into (5.8) the result follows. \square

5.1. Proof of Proposition 5.1. To finish the proof of Proposition 5.1, it remains to express Dz through the Hessian D^2u . We have from (4.8)

$$(5.12) \quad y = \varepsilon \kappa \frac{Du}{q+1},$$

$$(5.13) \quad Y^{n+1} = \varepsilon \left(1 - \frac{\kappa}{q+1} \right),$$

where $q = \sqrt{1 - \kappa(1 + |Du|^2)}$, see (2.3). From the definition of q we have $Dq = -\kappa Du D^2u / q$, thus

$$\begin{aligned} Dy &= \varepsilon \kappa \left[\text{Id} + \kappa \frac{Du \otimes Du}{q(q+1)} \right] \frac{D^2u}{q+1} \\ &= \varepsilon \kappa \mu_2 \frac{D^2u}{q+1}, \end{aligned}$$

where μ_2 is the matrix in (5.2). Now Lemma 5.1 yields

$$\begin{aligned} (5.14) \quad Dz &= \mu_1 \left(\text{Id} - y \otimes [y + Du Y^{n+1}] + t \varepsilon \kappa \mu_2 \frac{D^2u}{q+1} \right) \\ &= \mu_1 \mu_2 \left(\mu_2^{-1} \{ \text{Id} - y \otimes [y + Du Y^{n+1}] \} + t \varepsilon \kappa \frac{D^2u}{q+1} \right) \\ &= \mu_1 \mu_2 \left(\mu_2^{-1} \mathcal{M} + t \varepsilon \kappa \frac{D^2u}{q+1} \right) \end{aligned}$$

where $\mathcal{M} = \text{Id} - y \otimes [y + Du Y^{n+1}]$.

Using (5.12) we can further simplify the matrix $\mathcal{M} = \text{Id} - y \otimes [y + Du Y^{n+1}]$ to get

$$\begin{aligned} (5.15) \quad \mathcal{M} &= \text{Id} - y \otimes (y + Du Y^{n+1}) \\ &= \text{Id} - \frac{\varepsilon^2 \kappa^2}{(1+q)^2} Du \otimes Du - \frac{\varepsilon^2 \kappa (1 - \frac{\kappa}{1+q})}{1+q} Du \otimes Du \\ &= \text{Id} - \frac{\varepsilon^2 \kappa}{1+q} Du \otimes Du. \end{aligned}$$

By Lemma 4.2 we have for the inverse of μ_2 (see (5.2))

$$\begin{aligned} (5.16) \quad \mu_2^{-1} &= \text{Id} - \frac{\kappa}{q^2 + q + \kappa |Du|^2} Du \otimes Du \\ &= \text{Id} - \frac{\kappa}{1 - \kappa + q} Du \otimes Du, \end{aligned}$$

where the last equality follows from the definition of q , see (2.3). It remains to compute $\mu_2^{-1} \mathcal{M}$. From (5.16) and

(5.15) we obtain

$$\begin{aligned} \mu_2^{-1} \mathcal{M} &= \left[\text{Id} - \frac{\kappa}{1 - \kappa + q} Du \otimes Du \right] \left[\text{Id} - \frac{\varepsilon^2 \kappa}{1+q} Du \otimes Du \right] \\ &= \text{Id} + [I + II + III] Du \otimes Du \end{aligned}$$

where

$$\begin{aligned} I &= -\frac{\kappa}{1 - \kappa + q}, \\ II &= -\frac{\varepsilon^2 \kappa}{1+q}, \\ III &= \frac{\varepsilon^2 \kappa^2 |Du|^2}{(1+q)(1 - \kappa + q)}. \end{aligned}$$

It follows from (2.3) that $-\kappa|Du|^2 = q^2 - 1 + \kappa$, therefore

$$III = \frac{\varepsilon^2 \kappa (-q^2 + 1 - \kappa)}{(1+q)(1-\kappa+q)}.$$

Adding this to II we have

$$\begin{aligned} II + III &= \frac{\varepsilon^2 \kappa}{1+q} \left[-1 + \frac{-q^2 + 1 - \kappa}{1 - \kappa + q} \right] \\ &= -\frac{q\varepsilon^2 \kappa}{1 - \kappa + q}. \end{aligned}$$

Finally we compute the total sum

$$\begin{aligned} I + II + III &= -\frac{\kappa}{1 - \kappa + q} - \frac{q\varepsilon^2 \kappa}{1 - \kappa + q} \\ &= -\frac{\kappa}{1 - \kappa + q} [q\varepsilon^2 + 1] \\ &= -\frac{\kappa}{1 - \kappa + q} \left[\frac{q}{1 - \kappa} + 1 \right] \\ &= -\frac{\kappa}{1 - \kappa} \\ &= -\kappa\varepsilon^2, \end{aligned}$$

where the last line follows from the definition of κ , see (4.3).

Returning to (5.14) and utilising these computations we get

$$\begin{aligned} Dz &= \mu_1 \mu_2 \left[\mu_2^{-1} \mathcal{M} + t\varepsilon \kappa \frac{D^2 u}{q+1} \right] \\ &= \mu_1 \mu_2 \left[\text{Id} - \kappa\varepsilon^2 Du \otimes Du + t\varepsilon \kappa \frac{D^2 u}{q+1} \right]. \end{aligned}$$

This finishes the proof of Proposition 5.1. □

5.2. Proof of Theorem A. Now we are ready to finish the proof of Theorem A. Let $u \in C^2(\mathcal{U})$ be a solution to the refractor problem **(RP)** then from Proposition 5.1 we obtain

$$(5.17) \quad \det Dz = \det \mu_1 \det \mu_2 \left[\frac{t\varepsilon \kappa}{q+1} \right]^n \det \left[\frac{q+1}{t\varepsilon \kappa} \{ \text{Id} - \kappa\varepsilon^2 Du \otimes Du \} + D^2 u \right].$$

By Lemma 4.2 and (2.3) we have

$$\det \mu_2 = 1 + \frac{\kappa|Du|^2}{q(q+1)} = \frac{1 - \kappa + q}{q(q+1)}.$$

Similarly, we get

$$\det \mu_1 = \frac{\psi_{n+1}}{Y^{n+1}} \frac{1}{\nabla \psi \cdot Y}.$$

These in conjunction with (4.11) gives

$$\begin{aligned} \det \left[\frac{q+1}{t\varepsilon \kappa} \{ \text{Id} - \kappa\varepsilon^2 Du \otimes Du \} + D^2 u \right] &= \left[\frac{q+1}{t\varepsilon \kappa} \right]^n \frac{\det Dz}{\det \mu_1 \det \mu_2} \\ &= -\frac{f}{g} \frac{\psi_{n+1}}{|\nabla \psi|} \left[\frac{q+1}{t\varepsilon \kappa} \right]^n \frac{1}{\det \mu_1 \det \mu_2} \\ &= -(\nabla \psi \cdot Y) \frac{Y^{n+1}}{|\nabla \psi|} \frac{q(q+1)}{1 - \kappa + q} \left[\frac{q+1}{t\varepsilon \kappa} \right]^n \frac{f}{g}. \end{aligned}$$

Finally, recalling (4.8) and substituting the value of Y^{n+1} we see that

$$(5.18) \quad \det \left[\frac{q+1}{t\varepsilon\kappa} \{ \text{Id} - \kappa\varepsilon^2 Du \otimes Du \} + D^2 u \right] = -(\nabla\psi \cdot Y) \frac{Y^{n+1}}{|\nabla\psi|} \frac{q(q+1)}{1-\kappa+q} \left[\frac{q+1}{t\varepsilon\kappa} \right]^n \frac{f}{g} \\ = -\varepsilon q \left[\frac{q+1}{t\varepsilon\kappa} \right]^n \frac{\nabla\psi \cdot Y}{|\nabla\psi|} \frac{f}{g}$$

and the proof of Theorem A is now complete. \square

6. EXISTENCE OF SMOOTH SOLUTIONS

In this section we will have a provisional discussion on the existence of smooth solutions to (2.2). Our main objective is to apply the available regularity theory for the Monge-Ampère type equations, stemming from seminal paper [18], in order to establish the regularity of weak solutions of the refractor problem.

We first rewrite the equation (5.18) in a more concise form. Let us introduce the following matrix

$$(6.1) \quad G^{ij} = \frac{1}{t}(q+1)[\delta_{ij} - \kappa\varepsilon^2 u_i u_j].$$

Here $q = \sqrt{1 - \kappa(1 + |Du|^2)}$, see (2.3) and t is the stretch function determined from implicit equation $\psi(x + e_{n+1}u + tY) = 0$ as in Theorem A. Then the equation (5.18) transforms into

$$(6.2) \quad \det \left[-\frac{G}{\varepsilon\kappa} - D^2 u \right] = |h(x, u, Du)|, \quad \text{if } \kappa > 0, \varepsilon > 1 \quad u \in C^2(\mathcal{U}) \text{ and } -\frac{G}{\varepsilon\kappa} - D^2 u \geq 0,$$

$$(6.3) \quad \det \left[D^2 u + \frac{G}{\varepsilon\kappa} \right] = |h(x, u, Du)|, \quad \text{if } \kappa < 0, \varepsilon < 1 \quad u \in C^2(\mathcal{U}) \text{ and } \frac{G}{\varepsilon\kappa} + D^2 u \geq 0$$

with

$$(6.4) \quad h(x, u, Du) = -\varepsilon q \left[\frac{q+1}{t\varepsilon\kappa} \right]^n \frac{\nabla\psi \cdot Y}{|\nabla\psi|} \frac{f}{g}.$$

The existence of C^2 smooth solutions of (6.2) or (6.3) depend on the properties of the matrix G . Namely, it is shown in [18] that if we regard G as a function of variable $p = Du$ then the condition

$$(6.5) \quad -D_{p_k p_l}^2 G^{ij} \xi_i \xi_j \eta_k \eta_l \begin{cases} \leq -c_0 |\xi|^2 |\eta|^2 & \text{if } \kappa > 0 \\ \geq c_0 |\xi|^2 |\eta|^2 & \text{if } \kappa < 0 \end{cases} \quad \forall \xi, \eta \in \mathbb{R}^n, \xi \perp \eta,$$

with c_0 being a positive constant, is sufficient to obtain a priori $C^{1,1}$ bounds for the smooth solutions.

It is noteworthy to point out that the condition (6.5) and the C^2 estimates were derived in [18] for the Monge-Ampère type equations with variational structure emerging in optimal transport theory. The method used there is based on comparing the weak solution with the smooth one in a small ball. To employ this method successfully in the outset of refractor problem we need to establish a comparison principle, suitable mollification of the weak solution and a priori estimated for the smooth solutions of Dirichlet's problem in small balls.

The method outlined above gives the C^2 estimates for non-variational case as well, see [12, 13]. Therefore the local regularity result for the solutions to (6.2)-(6.3) with smooth w will follow once the matrix G verifies the condition (6.5). That done, the regularity of weak solutions reduces to the verification of the inequality (6.5) with some positive constant c_0 .

The conditions imposed on the matrix in (6.2)-(6.3) involving the Hessian implies that the Monge-Ampère equation is degenerate elliptic. The weak formulation of degenerate ellipticity will be discussed in Section 11. Postponing the precise definition of weak solutions until then we would like to point out how the ellipticity of equation follows if we consider those C^2 solutions of (6.2) (resp. (6.3)) for which at every point $x \in \mathcal{U}$ there is a hyperboloid (resp. ellipsoid) of revolution $H(\cdot, a, Z)$ touching u from above (reps. below) at x . Indeed, for

$H(x) = \ell_0 - \frac{a}{b} \sqrt{b^2 + |x - x_0|^2}$ the matrix $\mathcal{W}_H = -\frac{G}{\varepsilon\kappa} + D^2H$ is identically zero. To see this we consider the case of planar receiver Σ given as $X^{n+1} = m$ with $m > 0$. Without loss of generality we take $x_0 = 0$. Then $H(0) = \ell_0 - a$. On the other hand it follows from the definition of eccentricity $\varepsilon = \frac{\sqrt{a^2 + b^2}}{a}$ that $\ell_0 = m - a\varepsilon$, see Section 8. Next, a simple geometric reasoning yields the following explicit formula for the stretch function

$$(6.6) \quad t = \frac{m - H}{Y^{n+1}} = \frac{c + \frac{a}{b} \sqrt{b^2 + |x|^2}}{\varepsilon(1 - \frac{\kappa}{q+1})}.$$

We have $DH = -\frac{a}{b} \frac{x}{\sqrt{b^2 + |x|^2}}$. Consequently

$$(6.7) \quad D^2H = -\frac{a}{b\sqrt{b^2 + |x|^2}} \left(\text{Id} - \frac{x \otimes x}{b^2 + |x|^2} \right).$$

Moreover, recalling (4.3) we obtain $\kappa = 1 - \frac{1}{\varepsilon^2} = \frac{b^2}{c^2}$ where $c = \sqrt{a^2 + b^2}$. This gives

$$(6.8) \quad q(x) = \frac{1}{\varepsilon} \frac{b}{\sqrt{b^2 + |x|^2}},$$

in lieu of (2.3).

Thus combining these formulae for t and q we get from (6.1), (6.6) and (6.7)

$$\begin{aligned} \mathcal{W}_H &= -\frac{G}{\varepsilon\kappa} - D^2H \\ &= -\frac{q+1}{\kappa\varepsilon t} \left\{ [\delta_{ij} - \kappa\varepsilon^2 H_i H_j] + \frac{\kappa\varepsilon t D_{ij}^2 H}{q+1} \right\} \\ &= -\frac{q+1}{\kappa\varepsilon t} \left\{ \text{Id} - \underbrace{\left[\kappa\varepsilon^2 \frac{a^2}{b^2} \right] \frac{x \otimes x}{b^2 + |x|^2}}_{DH \otimes DH} + \frac{\kappa\varepsilon t}{q+1} \underbrace{\left[-\frac{a}{b\sqrt{b^2 + |x|^2}} \left(\text{Id} - \frac{x \otimes x}{b^2 + |x|^2} \right) \right]}_{D^2H} \right\}. \end{aligned}$$

From the definition of κ (1.7) it follows that $\kappa\varepsilon^2 \frac{a^2}{b^2} = \frac{b^2}{c^2} \varepsilon^2 \frac{a^2}{b^2} = 1$ implying

$$\mathcal{W}_H = -\frac{q+1}{\kappa\varepsilon t} \left(\text{Id} - \frac{x \otimes x}{b^2 + |x|^2} \right) \left\{ 1 - \frac{\kappa\varepsilon t}{q+1} \left[\frac{a}{b\sqrt{b^2 + |x|^2}} \right] \right\}.$$

Therefore, recalling (6.6) and (6.8) we easily compute

$$\begin{aligned} (6.9) \quad t &= \frac{c + \frac{a}{b} \sqrt{b^2 + |x|^2}}{\varepsilon(1 - \frac{\kappa}{q+1})} = (q+1) \frac{c + \frac{a}{b} \sqrt{b^2 + |x|^2}}{\varepsilon(q+1 - \kappa)} \\ &= (q+1) \frac{c + \frac{a}{b} \sqrt{b^2 + |x|^2}}{\varepsilon(q + \frac{1}{\varepsilon^2})} \\ &= \varepsilon(q+1) \frac{c + \frac{a}{b} \sqrt{b^2 + |x|^2}}{\varepsilon^2 q + 1} \\ &= \varepsilon(q+1) \sqrt{b^2 + |x|^2} \frac{c + \frac{a}{b} \sqrt{b^2 + |x|^2}}{\varepsilon b + \sqrt{b^2 + |x|^2}}. \end{aligned}$$

Returning to \mathcal{W}_H and utilizing (6.9) we obtain

$$\begin{aligned}
 1 - \frac{\kappa \varepsilon t}{q+1} \left[\frac{a}{b\sqrt{b^2 + |x|^2}} \right] &= 1 - \kappa \varepsilon^2 \frac{c + \frac{a}{b}\sqrt{b^2 + |x|^2}}{\sqrt{b^2 + |x|^2} + b\varepsilon} \\
 &= 1 - \varepsilon \kappa \frac{c}{a} \frac{a^2}{b^2} \\
 &= 1 - \varepsilon^2 \kappa \frac{a^2}{b^2} \\
 &= 0.
 \end{aligned}$$

A similar computation for the matrix $\mathcal{W}_E = \frac{G}{t\varepsilon|\kappa|} + D^2E$ can be carried out for the ellipsoids of revolution E (i.e. for $\varepsilon < 1, \kappa < 0$).

Since $-D^2u \geq -D^2H_{x_0}$ at x_0 and $\mathcal{W}_H = -\frac{G}{\varepsilon\kappa} - D^2H_{x_0} \equiv 0$ it follows that the equation $\det[-\frac{G}{\varepsilon\kappa} - D^2u] = h$ is degenerate elliptic.

Notice that for $\varepsilon < 1$ the weak solution has a supporting ellipsoid of revolution E_{x_0} at each point $x_0 \in \overline{U}$ touching Γ_u from below. In particular we see that if $u \in C^2$ then $Du = DE_{x_0}$, $-D^2u \leq -D^2E_{x_0}$ at x_0 . Thus $\frac{G}{\varepsilon\kappa} + D^2u \geq 0$ and we infer that (6.2) is degenerate elliptic. Analogously, using the hyperboloids as supporting functions, one can check that (6.3) is also degenerate elliptic.

7. PROOF OF THEOREM B: VERIFYING THE A3 CONDITION

In this section we explicitly compute the second derivatives in p variable of the matrix $G^{ij}(x, u, p)$ introduced in (6.1) where p is the dummy variable for Du . We will find a concise representation of the form $D_{p_k p_l} G^{ij} \xi^i \xi^j \eta^k \eta^l$ for $\xi, \eta \in \mathbb{R}^n, \xi \perp \eta$ and relate it with the second fundamental form of the receiver $\Sigma = \{Z \in \mathbb{R}^{n+1} : \psi(Z) = 0\}$ where $\psi : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ is a smooth function such that (2.4) holds. Recall that the existence of smooth solutions depends on the sign of the form $D_{p_k p_l} G^{ij} \xi^i \xi^j \eta^k \eta^l$, see (6.5) and the discussion in the previous section.

7.1. Computing the derivatives of stretch function t . Recall that by (4.10) $Z(x) = x + e_{n+1}u(x) + tY$. Differentiating $\psi(Z(x)) = 0$ with respect to p_k we get

$$(7.1) \quad \frac{t_{p_k}}{t} = -\frac{\sum_m \psi_m Y_{p_k}^m}{\sum_m \psi_m Y^m}, \quad k = 1, \dots, n.$$

After differentiating again by p_l we get

$$\begin{aligned}
 (7.2) \quad \frac{t_{p_k p_l}}{t} - \frac{t_{p_k} t_{p_l}}{t^2} &= - \left[\frac{\sum_{m,s} \psi_{ms} (Y_{p_l}^s t + Y^s t_{p_l}) Y_{p_k}^m + \sum \psi_m Y_{p_k p_l}^m}{(\nabla \psi \cdot Y)} \right. \\
 &\quad \left. - \frac{\sum_m \psi_m Y_{p_k}^m}{(\nabla \psi \cdot Y)^2} \left(\sum_{m,s} \psi_{ms} (Y_{p_l}^s t + Y^s t_{p_l}) Y^m + \sum_m \psi_m Y_{p_l}^m \right) \right] \\
 &= - \frac{1}{(\nabla \psi \cdot Y)} \left[\left(\nabla^2 \psi Y_{p_k} Y_{p_l} - \frac{\nabla \psi Y_{p_k}}{(\nabla \psi \cdot Y)} \nabla^2 \psi Y Y_{p_l} \right) t \right. \\
 &\quad \left. + \left(\nabla^2 \psi Y_{p_k} Y - \frac{\nabla \psi Y_{p_k}}{(\nabla \psi \cdot Y)} \nabla^2 \psi Y Y \right) t_{p_l} \right. \\
 &\quad \left. + \nabla \psi Y_{p_k p_l} - \frac{\nabla \psi Y_{p_k}}{(\nabla \psi \cdot Y)} \nabla \psi Y_{p_l} \right] = \\
 &= - \frac{1}{(\nabla \psi \cdot Y)} \left[\left(\nabla^2 \psi Y_{p_k} Y_{p_l} - \frac{\nabla \psi Y_{p_k}}{(\nabla \psi \cdot Y)} \nabla^2 \psi Y Y_{p_l} \right) t \right. \\
 &\quad \left. + \left(\nabla^2 \psi Y_{p_k} Y - \frac{\nabla \psi Y_{p_k}}{(\nabla \psi \cdot Y)} \nabla^2 \psi Y Y \right) t_{p_l} + \nabla \psi Y_{p_k p_l} \right] \\
 &\quad + \frac{t_{p_k} t_{p_l}}{t^2}.
 \end{aligned}$$

Rearranging the terms we infer

$$\begin{aligned}
 (7.3) \quad D_{p_k p_l}^2 \left(\frac{1}{t} \right) &= - \frac{t_{p_k p_l}}{t^2} + \frac{2 t_{p_k} t_{p_l}}{t^3} = \frac{1}{t(\nabla \psi \cdot Y)} \left[\left(\nabla^2 \psi Y_{p_k} Y_{p_l} - \frac{\nabla \psi Y_{p_k}}{(\nabla \psi \cdot Y)} \nabla^2 \psi Y Y_{p_l} \right) t \right. \\
 &\quad \left. + \left(\nabla^2 \psi Y_{p_k} Y - \frac{\nabla \psi Y_{p_k}}{\nabla \psi \cdot Y} \nabla^2 \psi Y Y \right) t_{p_l} + \nabla \psi Y_{p_k p_l} \right] \\
 &= \frac{1}{t(\nabla \psi \cdot Y)} \left[\frac{1}{t} \nabla^2 \psi Z_{p_k} Z_{p_l} + \nabla \psi Y_{p_k p_l} \right]
 \end{aligned}$$

where the last line follows from (7.1). Thus the second derivatives of $\frac{1}{t}$ can be computed from (7.3), while for the first order derivatives we have the formula (7.1).

Next, we want to compute the derivatives of $M_{ij} = (q+1)[\delta_{ij} - \kappa \varepsilon^2 p_i p_j]$ with respect to p . We have

$$D_{p_k} M_{ij} = q_{p_k} (\delta_{ij} - \kappa \varepsilon^2 p_i p_j) - (q+1) \kappa \varepsilon^2 [\delta_{kj} p_i + \delta_{ki} p_j].$$

The condition $\xi \perp \eta$ implies that the contribution of the terms involving δ_{kj} and δ_{ki} is zero. Thus we infer

$$D_{p_k p_l}^2 M_{ij} \xi^i \xi^j \eta^k \eta^l = q_{p_k p_l} \eta^k \eta^l [|\xi|^2 - \kappa \varepsilon^2 (p \cdot \xi)^2].$$

Recall that by definition $G = \frac{M}{t}$ hence from the product rule we have

$$\begin{aligned}
 (7.4) \quad D_{p_k p_l}^2 G^{ij} \xi^i \xi^j \eta^k \eta^l &= D_{p_k p_l}^2 \left(\frac{1}{t} \right) \eta^k \eta^l (q+1) [|\xi|^2 - \kappa \varepsilon^2 (p \cdot \xi)^2] \\
 &\quad + 2 \left(D_p \frac{1}{t} \cdot \eta \right) (D_p q \cdot \eta) [|\xi|^2 - \kappa \varepsilon^2 (p \cdot \xi)^2] \\
 &\quad + \frac{1}{t} (D_{p_k p_l}^2 q) \eta^k \eta^l [|\xi|^2 - \kappa \varepsilon^2 (p \cdot \xi)^2] \\
 &= S_{kl} \eta^k \eta^l [|\xi|^2 - \kappa \varepsilon^2 (p \cdot \xi)^2],
 \end{aligned}$$

where

$$(7.5) \quad S_{kl} = (q+1) D_{p_k p_l}^2 \left(\frac{1}{t} \right) + D_{p_k} \left(\frac{1}{t} \right) D_{p_l} q + D_{p_l} \left(\frac{1}{t} \right) D_{p_k} q + \frac{D_{p_k p_l}^2 q}{t}.$$

It follows from (4.8) that

$$(7.6) \quad Y_{p_k p_l}(q+1) + Y_{p_k} q_{p_l} + Y_{p_l} q_{p_k} + Y q_{p_k p_l} = e_{n+1} \varepsilon q_{p_k p_l}.$$

which after taking the inner product with $\nabla \psi$ and dividing the by $\nabla \psi \cdot Y$ yields

$$(7.7) \quad \begin{aligned} \frac{(q+1)\psi_{n+1}q_{p_k p_l}}{\nabla \psi \cdot Y} &= \frac{(q+1)\nabla \psi \cdot Y_{p_k p_l}}{\nabla \psi \cdot Y} + \frac{q_{p_k}\nabla \psi \cdot Y_{p_l}}{\nabla \psi \cdot Y} + \frac{q_{p_l}\nabla \psi \cdot Y_{p_k}}{\nabla \psi \cdot Y} + q_{p_k p_l} = \\ &= \frac{(q+1)\nabla \psi \cdot Y_{p_k p_l}}{\nabla \psi \cdot Y} + \frac{q_{p_k}\nabla \psi \cdot Y_{p_l}}{\nabla \psi \cdot Y} + \frac{q_{p_l}\nabla \psi \cdot Y_{p_k}}{\nabla \psi \cdot Y} + q_{p_k p_l} \\ &= \frac{(q+1)\nabla \psi \cdot Y_{p_k p_l}}{\nabla \psi \cdot Y} + tq_{p_k}D_{p_l}\left(\frac{1}{t}\right) + tq_{p_l}D_{p_k}\left(\frac{1}{t}\right) + q_{p_k p_l} \\ &= \frac{(q+1)\nabla \psi \cdot Y_{p_k p_l}}{\nabla \psi \cdot Y} + t\left(S_{kl} - (q+1)D_{p_k p_l}^2\left(\frac{1}{t}\right)\right). \end{aligned}$$

Consequently, with the aid of (7.3) we find that

$$\begin{aligned} S_{kl} &= \frac{q+1}{t(\nabla \psi \cdot Y)} (\psi_{n+1}q_{p_k p_l} - \nabla \psi \cdot Y_{p_k p_l}) + (q+1)D_{p_k p_l}^2\left(\frac{1}{t}\right) \\ &= \frac{q+1}{t(\nabla \psi \cdot Y)} (\psi_{n+1}q_{p_k p_l} - \nabla \psi \cdot Y_{p_k p_l}) + \frac{q+1}{t(\nabla \psi \cdot Y)} \left[\frac{1}{t}\nabla^2 \psi Z_{p_k} Z_{p_l} + \nabla \psi Y_{p_k p_l} \right] \\ &= \frac{q+1}{t(\nabla \psi \cdot Y)} \left[\frac{1}{t}\nabla^2 \psi Z_{p_k} Z_{p_l} + \psi_{n+1}q_{p_k p_l} \right]. \end{aligned}$$

It remains to recall that by (2.3)

$$(7.8) \quad q_{p_k p_l} = -\frac{\kappa}{q} \left[\delta_{kl} + \kappa \frac{p_k p_l}{q^2} \right]$$

and we conclude

$$(7.9) \quad D_{p_k p_l}^2 G^{ij} \xi^i \xi^j \eta^k \eta^l = \frac{q+1}{t(\nabla \psi \cdot Y)} \left[\frac{1}{t}\nabla^2 \psi Z_{p_k} Z_{p_l} + \psi_{n+1}q_{p_k p_l} \right] \eta^k \eta^l [|\xi|^2 - \kappa \varepsilon^2 (p \cdot \xi)^2].$$

It is worth noting that $|\xi|^2 - \kappa \varepsilon^2 (p \cdot \xi)^2$ is always positive. This is obvious if $\kappa < 0$. As for $\kappa > 0$ then we note that $|\xi|^2 - \kappa \varepsilon^2 (p \cdot \xi)^2 = |\xi|^2 \left(1 - \kappa \varepsilon^2 \left(p \cdot \frac{\xi}{|\xi|} \right)^2 \right) > 0$ in view of the estimate $|p| < \frac{1}{\sqrt{\varepsilon^2 - 1}}$, see Lemma 8.1. Furthermore, from (2.4) it follows that $D_{p_k p_l}^2 G^{ij}$ and \hat{S}_{kl} defined by

$$(7.10) \quad \hat{S}_{kl} = \frac{1}{t} \nabla^2 \psi Z_{p_k} Z_{p_l} + \psi_{n+1}q_{p_k p_l}$$

have the same signs. Thus it is enough to explore the form \hat{S}_{kl} instead.

7.2. Refining condition (6.5). Let Z_0 be a fixed point on Σ . Introduce a new coordinate system $\hat{x}_1, \dots, \hat{x}_n, \hat{x}_{n+1}$ near Z_0 , with \hat{x}_{n+1} having direction Y . Since (2.4) and (2.5) implies $\nabla \psi \neq 0$, without loss of generality we assume that near Z_0 , in $\hat{x}_1, \dots, \hat{x}_n, \hat{x}_{n+1}$ coordinate system Σ has a representation $\hat{x}_{n+1} = \varphi(\hat{x}_1, \dots, \hat{x}_n)$. Recall that the second fundamental form of Σ is

$$(7.11) \quad \Pi = \frac{\partial_{\hat{x}_i \hat{x}_j}^2 \varphi}{\sqrt{1 + |D\varphi|^2}}, \quad i, j = 1, \dots, n$$

if we choose the normal of Σ at Z_0 to be $\frac{(-D_{\hat{x}_1} \varphi, \dots, -D_{\hat{x}_n} \varphi, 1)}{\sqrt{1 + |D\varphi|^2}}$, $D\varphi = (D_{\hat{x}_1} \varphi, \dots, D_{\hat{x}_n} \varphi, 0)$.

Denote $\tilde{\psi}(Z) = Z^{n+1} - \varphi(z)$ and assume that near Z_0 , Σ is given by the equation $\tilde{\psi} = 0$. It follows that

$$(7.12) \quad \nabla^2 \tilde{\psi} = - \begin{vmatrix} \varphi_{11} & \cdots & \varphi_{1n} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \varphi_{n1} & \cdots & \varphi_{nn} & 0 \\ 0 & \cdots & 0 & 0 \end{vmatrix}.$$

Therefore for $Z = x + ue_{n+1} + tY$ we have $\nabla^2 \tilde{\psi} Y = 0$ and hence

$$(7.13) \quad \begin{aligned} \nabla^2 \tilde{\psi} Z_{p_k} Z_{p_l} &= \nabla^2 \tilde{\psi} (tY_{p_k} + t_{p_k} Y)(tY_{p_l} + t_{p_l} Y) \\ &= t^2 \nabla^2 \tilde{\psi} Y_{p_k} Y_{p_l} \\ &= -t^2 \nabla^2 \varphi Y_{p_k} Y_{p_l}. \end{aligned}$$

By (5.12) $Y(q+1) = (\varepsilon \kappa p, \varepsilon q + \varepsilon - \kappa)$ where $y(q+1) = \varepsilon \kappa p$. Differentiating this equality with respect to p_k we infer

$$(7.14) \quad Y_{p_k}(q+1) + Y q_{p_k} = \varepsilon(\kappa \hat{e}_k + q_{p_k} \hat{e}_{n+1})$$

hence

$$(7.15) \quad Y_{p_k} = \frac{1}{q+1} [-Y q_{p_k} + \varepsilon(\kappa \hat{e}_k + q_{p_k} \hat{e}_{n+1})].$$

On the other hand (7.12) and $\hat{e}_{n+1} = Y$ yield

$$(7.16) \quad \begin{aligned} \nabla^2 \tilde{\psi} Y_{p_k} &= \frac{1}{q+1} \nabla^2 \tilde{\psi} [-Y q_{p_k} + \varepsilon(\kappa \hat{e}_k + q_{p_k} \hat{e}_{n+1})] \\ &= \frac{\varepsilon}{q+1} \nabla^2 \tilde{\psi} (\kappa \hat{e}_k + q_{p_k} \hat{e}_{n+1}) \\ &= \frac{\varepsilon \kappa}{q+1} \nabla^2 \tilde{\psi} \hat{e}_k. \end{aligned}$$

Since $\nabla^2 \tilde{\psi}$ is symmetric we infer

$$(7.17) \quad \nabla^2 \tilde{\psi} Y_{p_k} Y_{p_l} = \frac{\varepsilon^2 \kappa^2}{(q+1)^2} \nabla^2 \tilde{\psi} \hat{e}_k \hat{e}_l.$$

Plugging (7.17) into (7.13) we finally obtain

$$(7.18) \quad \begin{aligned} \nabla^2 \tilde{\psi} Z_{p_k} Z_{p_l} &= -t^2 \frac{\varepsilon^2 \kappa^2}{(q+1)^2} \nabla^2 \varphi \hat{e}_k \hat{e}_l \\ &= -t^2 \frac{\varepsilon^2 \kappa^2}{(q+1)^2} \sqrt{1 + |D\varphi|^2} \text{II} \end{aligned}$$

where II is the second fundamental form of Σ at Z_0 , see (7.11). This in conjunction with (7.8) yields

$$(7.19) \quad \hat{S}_{kl} = - \left[\frac{1}{t} \left(\frac{t\varepsilon\kappa}{q+1} \right)^2 \sqrt{1 + |D\varphi|^2} \text{II} + \frac{\kappa}{q} (\text{Id} + \kappa \frac{p \otimes p}{q^2}) \right].$$

Summarizing

(7.20)

$$D_{p_k p_l}^2 G^{ij} \xi^i \xi^j \eta^k \eta^l = -\frac{q+1}{t(\nabla \psi \cdot Y)} \left[\frac{1}{t} \left(\frac{t\varepsilon\kappa}{q+1} \right)^2 \sqrt{1+|D\varphi|^2} \text{II} + \frac{\kappa}{q} (\text{Id} + \kappa \frac{p \otimes p}{q^2}) \right] \eta^k \eta^l [|\xi|^2 - \kappa \varepsilon^2 (p \cdot \xi)^2].$$

7.3. Examples of receivers. Let us consider the case of horizontal receiver $Z^{n+1} = m > 0$ for some positive number m . Then $\text{II} = 0$ implying that

$$D_{p_k p_l}^2 G^{ij} \xi^i \xi^j \eta^k \eta^l = -\frac{q+1}{t(\nabla \psi \cdot Y)} \frac{\kappa}{q} \left(\text{Id} + \kappa \frac{p \otimes p}{q^2} \right) \eta^k \eta^l [|\xi|^2 - \kappa \varepsilon^2 (p \cdot \xi)^2].$$

If $\kappa > 0$ then in view of (8.5) we have $|\xi|^2 - \kappa \varepsilon^2 (p \cdot \xi)^2 > 0$ and clearly $D_{p_k p_l}^2 G^{ij} \xi^i \xi^j \eta^k \eta^l < 0$. Hence (6.5) is not satisfied for this case.

As for $\kappa < 0$ we compute

$$(7.21) \quad -\frac{\kappa}{q} (|\eta|^2 + \kappa \frac{(p \cdot \eta)^2}{q^2}) = \frac{|\kappa||\eta|^2}{q} (1 - \frac{|\kappa|(p \cdot \eta)^2}{q^2 |\eta|^2}) \geq \frac{|\kappa||\eta|^2}{q} (1 - \frac{|\kappa|p|^2}{q^2}) = \frac{|\kappa|(1+|\kappa|)}{q^3} |\eta|^2 \geq c_0 |\eta|^2$$

where $c_0 > 0$ depends only on $\sup |p|$ and ε . Consequently, $D_{p_k p_l}^2 G^{ij} \xi^i \xi^j \eta^k \eta^l > 0$ and (6.5) is not true for horizontal receivers $Z^{n+1} = m > 0$.

Next we construct examples of ψ satisfying (6.5).

Case 1: $\varepsilon > 1, \kappa > 0$.

As we have seen above $|\xi|^2 - \kappa \varepsilon^2 (p \cdot \xi)^2 > 0$. Therefore in order to get $D_{p_k p_l}^2 G^{ij} \xi^i \xi^j \eta^k \eta^l > 0$ we must assume that $\text{II} < -\delta \text{Id}$ for some constant $\delta > 0$. It is enough to show that

$$J := \frac{1}{t} \left(\frac{t\varepsilon\kappa}{q+1} \right)^2 \sqrt{1+|D\varphi|^2} \text{II} + \frac{\kappa}{q} (\text{Id} + \kappa \frac{p \otimes p}{q^2}) < 0.$$

We have

$$(7.22) \quad \frac{\kappa}{q} (|\eta|^2 + \kappa \frac{(p \cdot \eta)^2}{q^2}) \leq |\eta|^2 \frac{\kappa}{q} (1 + \kappa \frac{|p|^2}{q^2}) = |\eta|^2 \frac{\kappa}{q^3} (1 - \kappa).$$

Therefore using the fact that $q < 1$ for $\kappa > 0$ we get

$$(7.23) \quad \begin{aligned} J &\leq -\delta t \left(\frac{\varepsilon\kappa}{q+1} \right)^2 \text{Id} + \frac{\kappa}{q^3} (1 - \kappa) \text{Id} \\ &= -\kappa \text{Id} \left(t\delta \frac{\varepsilon^2 \kappa}{(1+q)^2} - \frac{1}{\varepsilon^2} \right) \\ &\leq -\kappa \text{Id} \left(t\delta \frac{\varepsilon^2 \kappa}{4} - \frac{1}{\varepsilon^2} \right) \\ &< 0 \end{aligned}$$

if $t > \frac{4}{\delta \varepsilon^4 \kappa}$. Thus if Σ is strictly concave and is sufficiently far from \mathcal{U} then (6.5) is satisfied for $\varepsilon > 1, \kappa > 0$.

Case 2: $\varepsilon < 1, \kappa < 0$.

Again, we obviously have $|\xi|^2 - \kappa\varepsilon^2(p \cdot \xi)^2 > 0$. Therefore we demand $J > 0$. For this we suppose that $\Pi > \delta \text{Id}$ for some $\delta > 0$. Then by (7.21)

$$\begin{aligned}
 (7.24) \quad J &\geq \delta t \left(\frac{\varepsilon \kappa}{q+1} \right)^2 \text{Id} - \frac{|\kappa|}{q} \text{Id} = \\
 &= |\kappa| \text{Id} \left(\delta t \frac{\varepsilon^2 |\kappa|}{(1+q)^2} - \frac{1}{q} \right) \quad (\text{recall that } q = \sqrt{1 + |\kappa|(1 + |p|^2)} > 1) \\
 &= |\kappa| \text{Id} \left(\delta t \frac{\varepsilon^2 |\kappa|}{(1+q)^2} - 1 \right) \\
 &\geq |\kappa| \text{Id} \left(\delta t \frac{\varepsilon^2 |\kappa|}{(1 + \sqrt{1 + |\kappa|(1 + \mu^2)})^2} - 1 \right) \\
 &> 0
 \end{aligned}$$

if $t > \frac{(1 + \sqrt{1 + |\kappa|(1 + \mu^2)})^2}{\delta |\kappa| \varepsilon^2}$ and $|p| \leq \mu$, where μ depends on the Lipschitz constant of the solution, see (8.5). Thus if Σ is strictly convex and is sufficiently far from \mathcal{U} then (6.5) is satisfied for $\varepsilon < 1, \kappa < 0$.

Remark 7.1. We summarize the following conclusions based on the discussion in this section:

- The computation above shows that (6.5) is true if $\kappa > 0, \Pi > 0$ or if $\kappa < 0, \Pi < 0$ and $\text{dist}(\mathcal{U}, \mathcal{V})$ is large enough.
- If $k > 0$ and Σ is a graph of strictly concave function $Z^{n+1} = \varphi(z)$ then (2.4) holds. Hence the refracted rays intersect Σ at a unique point.
- We can extend Σ to entire space such that the resultd surface is still concave if say $\kappa > 0$, hence without loss of generality we can assume that Σ is entire concave surface and so is $\Sigma + Me_{n+1}$, for $M \gg 1$. We will take advantage of this in Lemmas 8.3 and 10.2

8. ADMISSIBLE FUNCTIONS

The refractive properties of ellipses and hyperbolas have been known since ancient times [19]. Furthermore, hyperboloids and ellipsoids of revolution share the same properties. This section is devoted to the class of functions obtained as envelopes of halves of ellipsoids and hyperboloids of revolution.

8.1. Ellipsoids. Throughout this paper by ellipsoid we mean the lower half of an ellipsoid of revolution with focal axis parallel to e_{n+1} . Such surface can be regarded as the graph of

$$(8.1) \quad E(x, a, Z) = Z^{n+1} - a\varepsilon - a\sqrt{1 - \frac{(x-z)^2}{a^2(1-\varepsilon^2)}}$$

where a is the larger semiaxis, ε - the eccentricity, and Z the higher focus, see Figure 2. Moreover we have that

$$(8.2) \quad DE = \frac{1}{a(1-\varepsilon^2)} \frac{x-z}{\sqrt{1 - \frac{(x-z)^2}{a^2(1-\varepsilon^2)}}}.$$

Notice that at the points x where $|x-z| = a\sqrt{1-\varepsilon^2}$ the gradient $|DE|$ is unbounded.

8.2. Hyperboloids. It is convenient to introduce the lower sheet of hyperboloids of revolution

$$(8.3) \quad H(x, a, Z) = Z^{n+1} - a\varepsilon - a\sqrt{1 + \frac{(x-z)^2}{a^2(\varepsilon^2-1)}}$$

where a is the larger semiaxis, ε the eccentricity, and Z the upper focus, see Figure 2. Differentiating H we obtain

$$(8.4) \quad DH = -\frac{1}{(\varepsilon^2-1)} \frac{x-z}{\sqrt{a^2 + \frac{(x-z)^2}{\varepsilon^2-1}}}.$$

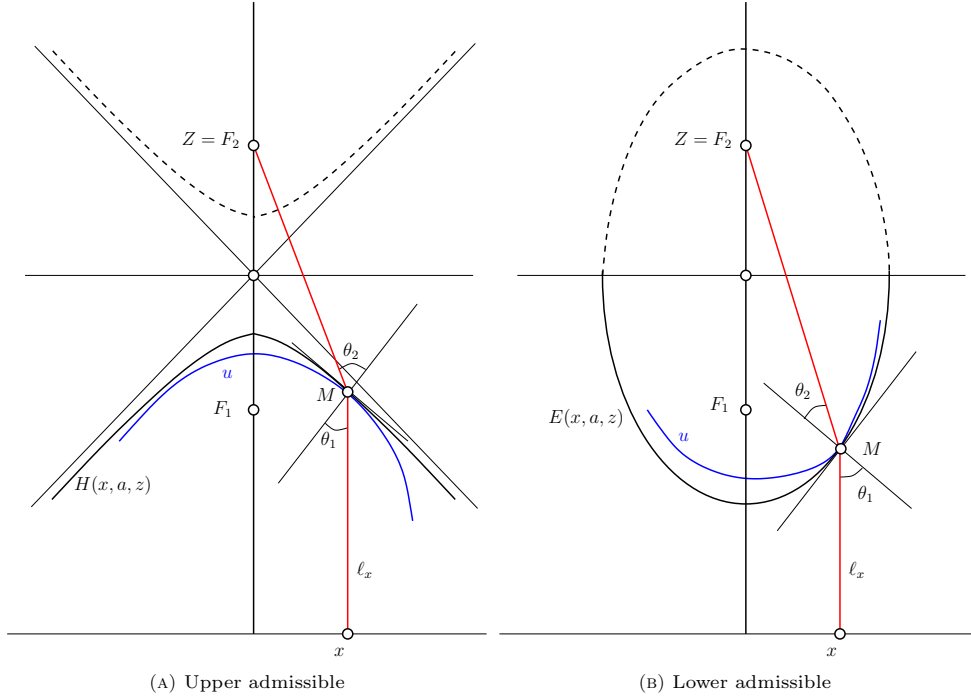


FIGURE 2. Supporting surfaces

8.3. Supporting hyperboloids.

Definition 8.1. A function $u : \mathcal{U} \rightarrow \mathbb{R}$ is said to be upper (resp. lower) admissible if for any $x_0 \in \mathcal{U}$ there is $Z \in \mathcal{V}$ and $a > 0$ such that $H(x_0, a, Z) = u(x_0)$ (resp. $E(x_0, a, Z) = u(x_0)$) and $H(x, a, Z) \geq u(x)$, $x \in \mathcal{U}$ (resp. $E(x, a, Z) = u(x)$). H (resp. E) is called a supporting function of u at x_0 . The class of all upper admissible functions is denoted by $\overline{\mathbb{W}}_H(\mathcal{U}, \mathcal{V})$ (resp. $\underline{\mathbb{W}}_E(\mathcal{U}, \mathcal{V})$).

In what follows we focus on upper admissible functions, the lower admissible functions can be studied in similar fashion. If the generalisation is not straightforward then we will outline the proof.

Formula (8.4) yields uniform Lipschitz estimates for $\overline{\mathbb{W}}_H(\mathcal{U}, \mathcal{V})$.

Lemma 8.1. Let $\mathbb{H}_{a_0}^+(\mathcal{U}, \mathcal{V})$ be the set of all hyperboloids $H(x, a, Z) \geq 0$, $x \in \mathcal{U}$ for some $Z \in \mathcal{V}$ such that $a > a_0$ for some fixed $a_0 \geq 0$. Then

$$(8.5) \quad \sup_{\mathbb{H}_{a_0}^+(\mathcal{U}, \mathcal{V})} \|DH\|_\infty \leq \frac{1}{\sqrt{\varepsilon^2 - 1}} \frac{d_0}{\sqrt{a_0^2(\varepsilon^2 - 1) + d_0^2}} \quad (\text{if } a_0 > 0)$$

where $d_0 = \sup_{x \in \mathcal{U}, z \in \widehat{\mathcal{V}}} |x - z|$. Furthermore,

$$\|Du\|_{L^\infty(\mathcal{U}')} < \frac{1}{\sqrt{\varepsilon^2 - 1}}, \quad \forall \mathcal{U}' \subset \subset \mathcal{U}, u \in \overline{\mathbb{W}}_H(\mathcal{U}, \mathcal{V}).$$

Proof. From (8.4) we have for $x \neq z$

$$\begin{aligned}
 (8.6) \quad |DH| &= \frac{1}{(\varepsilon^2 - 1)} \frac{|x - z|}{\sqrt{a^2 + \frac{|x - z|^2}{\varepsilon^2 - 1}}} \\
 &= \frac{1}{\sqrt{\varepsilon^2 - 1}} \frac{|x - z|}{\sqrt{a^2(\varepsilon^2 - 1) + |x - z|^2}} = \frac{1}{\sqrt{\varepsilon^2 - 1}} \frac{1}{\sqrt{\frac{a^2(\varepsilon^2 - 1)}{|x - z|^2} + 1}} \\
 &\leq \frac{1}{\sqrt{\varepsilon^2 - 1}} \frac{1}{\sqrt{\frac{a^2(\varepsilon^2 - 1)}{\sup_{x \in \mathcal{U}, z \in \hat{\mathcal{V}}} |x - z|^2} + 1}}.
 \end{aligned}$$

If $a \geq a_0 > 0$ then the first inequality immediately follows.

In order to prove the second inequality let us suppose that for some fixed subdomain $\mathcal{U}' \subset \subset \mathcal{U}$ there are $p_k \in \partial u(x_k)$ such that $x_k \in \mathcal{U}'$, $p_k \rightarrow p_0$, $x_k \rightarrow x_0$ and $|p_0| = \frac{1}{\sqrt{\varepsilon^2 - 1}}$. Here $\partial u(x)$ is the subdifferential of u at x . It is clear that $p_0 \in \partial u(x_0)$ and hence there is a supporting hyperplane for u at x_0 with slope p_0 . If u is strictly concave at $x_0 \in U$ then near x_0 one can find \bar{x}_0 such that there is $\bar{p}_0 \in \partial u(\bar{x}_0)$ with $|\bar{p}_0| > \frac{1}{\sqrt{\varepsilon^2 - 1}}$ which is in contradiction with the first inequality. Thus suppose that there is a straight segment in the graph of u passing through $(x_0, u(x_0))$. But this is impossible because u is admissible and therefore Γ_u cannot contain straight segments. \square

Lemma 8.2. *Let $\{u_k\}$ be a sequence of upper admissible function such that $u_k \rightarrow u_0$ uniformly in \mathcal{U} . If $x_k \in \mathcal{U}$, $x_k \rightarrow x_0$ and H_k are supporting functions of u_k at x_k then u_0 has an upper supporting function H_0 at x_0 and $H_k \rightarrow H_0$ uniformly in \mathcal{U} .*

Proof. One way to check the claim is to use some well known fact from convex analysis. Consider the convex sets $G_k = \{X \in \mathbb{R}^{n+1} : x \in \mathcal{U}, 0 < u_k(x) < X^{n+1}\}$ and $\mathcal{H}_k = \{X \in \mathbb{R}^{n+1} : x \in \mathcal{U}, 0 < H_k(x) < X^{n+1}\}$ where $x = \hat{X}$. Then $\mathcal{H}_k \subset G_k$ and $(x_k, u_k(x_k)) \in \partial \mathcal{H}_k \cap \partial G_k$. Thus, from uniform convergence $u_k \rightarrow u_0$ we infer that the limit set $G_0 = \{X \in \mathbb{R}^{n+1} : x \in \mathcal{U}, 0 < u_0(x) < X^{n+1}\}$ is a subset of $\mathcal{H}_0 = \{X \in \mathbb{R}^{n+1} : x \in \mathcal{U}, 0 < H_0(x) < X^{n+1}\}$, see [1] Chapter 5.2. Furthermore, from $x_k \rightarrow x_0 \in \bar{\mathcal{U}}$ it follows that there is $X_0 \in \partial G_0 \cap \partial \mathcal{H}_0$ such that $\hat{X}_0 = x_0$. Therefore we conclude that H_0 is a supporting hyperboloid of u_0 at x_0 . \square

8.4. Continuous expansion of hyperboloids. If $u \in \overline{\mathbb{W}}_H(\mathcal{U}, \Sigma)$ then it turns out that u is also admissible with respect with $\tilde{\Sigma}$, the receiver moved vertically upwards in e_{n+1} direction. In other words, the same admissible u will be R -convex with respect to a family of surfaces obtained from Σ by translation in e_{n+1} direction. We will need this observation in order to construct smooth solutions of our problem in small balls, see Section 14.

Lemma 8.3. *Let $\tilde{\Sigma} = \Sigma + M e_{n+1}$ for some $M > 0$.*

- (i) *For any fixed x_0 and $H_1(x) = H(x, a_1, Z_1) \in \mathbb{H}(\mathcal{U}, \Sigma)$ there is $H_2(x) = H(x, a_2, Z_2)$ with $Z_2 \in \tilde{\Sigma}$ and touching H_1 from above at x_0 .*
- (ii) *In particular if $u \in \overline{\mathbb{W}}_H(\mathcal{U}, \Sigma)$ then also $u \in \overline{\mathbb{W}}_H(\mathcal{U}, \tilde{\Sigma})$.*

Proof. (i) Let $\xi_1 = H_1(x_0)$ and $X_0 = (x_0, \xi_1)$. For $s > 1$ we consider $Z_2 = X_0 + s(Z_1 - X_0)$. By construction X_0, Z_1 and Z_2 lie on the same line. To determine a_2 we utilize two geometric properties of hyperbola, namely that the difference of distances of X_0 from Z_2 and the lower focus Z'_2 is $2a_2$ and $|X_0 Z'_2| = \varepsilon |X_0 D|$ where $|X_0 D|$ is the distance of X_0 from the lower directrix $X^{n+1} = Z^{n+1} - a_2 \varepsilon - a_2 / \varepsilon$. Therefore if P is on the graph of H_2 we get that $|P Z_2| = 2a_2 + |P Z'_2| = -a_2(\varepsilon^2 - 1) + s\varepsilon(Z_1^{n+1} - \xi_1)$. Taking $P = X_0$ in this equation $|P Z_2| = \varepsilon |Z_1 - X_0|$ one finds that

$$(8.7) \quad a_2 = \frac{1}{\varepsilon^2 - 1} [s\varepsilon(Z_1^{n+1} - \xi_1) - |s(Z_1 - X_0)|].$$

As for (ii), we choose $s_0 > 1$ so that $X_0 + s(Z_1 - X_0) \in \tilde{\Sigma}$. Consequently from (i) it follows that $Z_2 = X_0 + s_0(Z_1 - X_0)$ is the focus of supporting hyperboloid $H(\cdot, a_2, Z_2)$ at x_0 where a_2 is given by (8.7). Therefore $u \in \overline{\mathbb{W}}_H(\mathcal{U}, \tilde{\Sigma})$. \square

9. B-TYPE WEAK SOLUTIONS: PROOF OF THEOREM C1

In this section we introduce our first notion of weak solution for the refractor problem **(RP)**. For any upper admissible function $u \in \overline{\mathbb{W}}_H(\mathcal{U}, \mathcal{V})$ we define the mapping $\mathcal{S}_u : \mathcal{V} \rightarrow \mathcal{U}$ as follows

$$\mathcal{S}_u(Z) = \{x \in \mathcal{U} : \exists \text{ a supporting hyperboloid of } u \text{ at } x \text{ with focus at } Z \in \mathcal{V}\}.$$

For any Borel set $\omega \subset \mathcal{V}$ we put

$$(9.1) \quad \mathcal{S}_u(\omega) = \bigcup_{Z \in \omega} \mathcal{S}_u(Z).$$

We will write $\mathcal{S}(E)$ instead of $\mathcal{S}_u(E)$ if there is no confusion.

Proposition 9.1. *For $u \in \overline{\mathbb{W}}_H(\mathcal{U}, \mathcal{V})$ the corresponding mapping \mathcal{S} enjoys the following properties:*

- a) $\mathcal{S} : \mathcal{V} \rightarrow \Pi$ maps the closed sets to closed sets.
- b) The mapping \mathcal{S} is one-to-one modulo a set of vanishing measure, i.e.

$$|\{x \in \Pi : x \in \mathcal{S}(Z_1) \cap \mathcal{S}(Z_2) \text{ for } Z_1 \neq Z_2, \ Z_i \in \mathcal{V}, i = 1, 2\}| = 0.$$

- c) The family $\mathcal{F} = \{E \subset \mathcal{V} \text{ such that } \mathcal{S}(E) \text{ is measurable}\}$ is σ -algebra.

Proof. The first claim a) follows directly from Lemma 8.2.

In order to prove b) we set $A = \{x \in \Pi : x \in \mathcal{S}(Z_1) \cap \mathcal{S}(Z_2) \text{ for } Z_1 \neq Z_2, \ Z_i \in \mathcal{V}, i = 1, 2\}$. If $x \in A$ then u cannot be differentiable at x thanks to (2.4). Notice that if Σ is a strictly concave graph over the plane $\{x_{n+1} = 0\}$ then (2.4) is satisfied, see Remark 7.1. By Aleksandrov's theorem the concave function u is twice differentiable a.e. Hence $|A| = 0$.

As for c) we must check that the following three conditions hold, see e.g. [2]

- 1) $\mathcal{V} \in \mathcal{F}$,
- 2) if $A \in \mathcal{F}$ then $\mathcal{V} \setminus A \in \mathcal{F}$,
- 3) if $A_i \in \mathcal{F}$ then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

We first prove 1). If $A_i \in \mathcal{V}$ is any sequence of subsets of \mathcal{V} then clearly $\mathcal{S}(\bigcup_{i=1}^{\infty} A_i) = \bigcup_{i=1}^{\infty} \mathcal{S}(A_i)$. Writing $\mathcal{V} = \bigcup_{i=1}^{\infty} E_i$, where $E_i \subset \mathcal{V}$ are closed subsets we conclude that $\mathcal{S}(\mathcal{V}) = \mathcal{S}(\bigcup_{i=1}^{\infty} E_i) = \bigcup_{i=1}^{\infty} \mathcal{S}(E_i)$. From a) it follows that $\mathcal{S}(E_i)$ is closed for any i , and hence measurable, implying that $\mathcal{S}(\mathcal{V})$ is measurable.

2) Let $A \in \mathcal{F}$. We use the following elementary identity

$$(9.2) \quad \mathcal{S}(\mathcal{V} \setminus A) = [\mathcal{S}(\mathcal{V}) \setminus \mathcal{S}(A)] \bigcup [\mathcal{S}(\mathcal{V} \setminus A) \cap \mathcal{S}(A)].$$

From b) it follows that $|\mathcal{S}(\mathcal{V} \setminus A) \cap \mathcal{S}(A)| = 0$. Therefore $|\mathcal{S}(\mathcal{V} \setminus A)| = |\mathcal{S}(\mathcal{V}) \setminus \mathcal{S}(A)|$ and 2) is proven.

It remains to check 3). Without loss of generality we assume that A_i 's are disjoint, see [2]. Thus, letting $A_i \in \mathcal{F}$, $A_i \cap A_j = \emptyset$, $i \neq j$ we get

$$\begin{aligned} \sum_{i=1}^{\infty} |\mathcal{S}(A_i)| &\geq |\mathcal{S}(\cup_{i=1}^{\infty} A_i)| \geq \\ &\geq \sum_{i=1}^{\infty} |\mathcal{S}(A_i)| - \sum_{i,j=1}^{\infty} |\mathcal{S}(A_i) \cap \mathcal{S}(A_j)| \geq \\ &\geq \sum_{i=1}^{\infty} |\mathcal{S}(A_i)|. \end{aligned}$$

□

For a given function $u \in \mathbb{W}_H(\mathcal{U}, \mathcal{V})$ we consider the set function

$$(9.3) \quad \beta_u(\omega) = \int_{\mathcal{S}(\omega)} f$$

where $\omega \subset \mathcal{V}$ is a Borel subset. Since \mathcal{F} contains the closed sets (see part a) above) we infer that $\beta_{u,f}$ is a Borel measure. Moreover, from the proof of Proposition 9.1 b) it follows that $\beta_{u,f}$ is countably additive.

Definition 9.1. A function u (or its graph Γ_u) is said to be a *B-type weak solution to (RP)* if $u \in \overline{\mathbb{W}}_H(\mathcal{U}, \mathcal{V})$ and the following two identities holds

$$(9.4) \quad \begin{cases} \beta_{u,f}(\omega) = \int_{\omega} g d\mathcal{H}^n, \text{ for any Borel set } \omega \subset \mathcal{V} & \text{and} \\ \mathcal{S}_u(\mathcal{V}) = \mathcal{U}. \end{cases}$$

9.1. Existence of weak solutions of B-type. The measure β , defined in (9.3) is weakly continuous. We have

Lemma 9.1. Let u_k be a sequence of B-type weak solutions in the sense of Definition 9.3 and β_k is the associated measure, defined by (9.3). If $u_k \rightarrow u$ uniformly on compact subsets of \mathcal{U} then u is R -concave and β_k weakly converges to $\beta_{u,f}$.

Proof. Once the σ additivity is established then the proof of lemma is standard, see for the classical case [22] pp 14-18, [5] and [6], [11] for refractor and reflector problems respectively. The proof for supporting ellipsoids is carried out also [7]. That u is admissible follows from Lemma 8.2. Recall that the weak convergence is equivalent to the following two inequalities (see [2] Theorem 4.5.1)

- 1) $\limsup_{k \rightarrow \infty} \beta_k(E) \leq \beta(E)$ for any compact $E \subset \mathcal{V}$,
- 2) $\liminf_{k \rightarrow \infty} \beta_k(J) \geq \beta(J)$ for any open $J \subset \mathcal{V}$.

Take a closed set E and let E_δ^* be an δ -neighbourhood of the closed set $E^* = \mathcal{S}(E)$, see Lemma 9.1 a). We claim that for any $\delta > 0$ there is $i_0 \in \mathbb{N}$ such that $\mathcal{S}_i(E) \subset E_\delta^*$ whenever $i > i_0$, where \mathcal{S}_i is the mapping corresponding to u_i . If this fails then there is $\delta > 0$ and a sequence of points $x_i \in \mathcal{S}_i(E)$ such that $x_i \in \mathbb{C}E_\delta^*$. By definition there is $Z_i \in E$ such that $x_i \in \mathcal{S}_i(Z_i)$. Suppose that $x_i \rightarrow x_0$, for some x_0 , and $Z_i \rightarrow Z_0 \in E$ at least for a subsequence. Thus, $x_0 \in \mathbb{C}E_\delta^*$, $x_0 \in \mathcal{S}(Z_0)$ and $Z_0 \in E$ which is a contradiction.

To prove the second inequality we let $J \subset \mathcal{V}$ be an open subset and denote $J^* = \mathcal{S}(H)$. By Lemma 9.1 c) J^* is measurable, hence for any small $\delta > 0$ there is a closed set J_δ^* such that $J_\delta^* \subset J^*$ and $|J^*| - \delta \leq |J_\delta^*| \leq |J^*|$. This is possible because by Proposition 9.1 b) \mathcal{S} is one-to-one modulo a set of measure zero. Let N_δ be an open set, $|N_\delta| < \delta$ containing the points where the inverse of \mathcal{S} is not defined. We claim that there is k_0 such that

$$(9.5) \quad J_\delta^* \setminus N_\delta \subset J_k^* \stackrel{def}{=} \mathcal{S}_k(J), \quad \text{for any } k \geq k_0.$$

Here \mathcal{S}_k is the mapping generated by u_k . Proof of (9.5) is by contradiction. If (9.5) fails then there is $x_k \in J_\delta^* \setminus N_\delta$ and $x_k \notin J_k^*$. We can assume that $x_k \rightarrow x_0$. Since $J_\delta^* \setminus N_\delta$ is closed it follows that $x_0 \in J_\delta^* \setminus N_\delta$. By definition of N_δ the inverse of \mathcal{S} is one-to-one on $J_\delta^* \setminus N_\delta$. Thus there is a unique $Z_0 \in H$ such that $x_0 = \mathcal{S}(Z_0)$. Furthermore, there is an open neighborhood of Z_0 contained in J because J is open. If $H(x, \sigma_k, Z_k)$ is a supporting hyperboloid of u_k at x_k it follows from Lemma 8.2 that $x_k \in \mathcal{S}_k(Z_k)$, $Z_k \rightarrow Z_0$. Thus for large k , $\{Z_k\}$ is in some neighborhood of $Z_0 \in J$ implying that $x_k \in J_k^*$ which contradicts our supposition. \square

Proposition 9.2. *Let $f : \mathcal{U} \rightarrow \mathbb{R}$ and $g : \mathcal{V} \rightarrow \mathbb{R}$ be two nonnegative integrable functions. If $\mathcal{U} \subset \Pi$ and $\mathcal{V} \subset \Sigma$ are bounded domains such that the energy balance condition (1.3) and (2.11) hold then there exists a B -type weak solution to the problem (RP).*

Notice that we do not exclude the case $\mathcal{U} \cap \widehat{\mathcal{V}} \neq \emptyset$.

Proof. The proof of Proposition 9.2 is by approximation argument, see [7], [11], [27]. Let $g_N = \sum_{i=1}^N C_i \delta_{Z_i}$ with $C_i \geq 0$ such that $\sum_{i=1}^N C_i = \int_{\mathcal{U}} f(x) dx$, $Z_i \in \Sigma$ and δ_{Z_i} are atomic measures supported at Z_i . For each g_N we construct a B -type solution u_N . Then sending $N \rightarrow \infty$ and using the compactness argument together with weak convergence of g_N to g , Lemma 9.1, one will arrive at desired result.

First, for each $Z \in \Sigma$ we define

$$(9.6) \quad \bar{a}(Z) = \frac{\varepsilon Z^{n+1} - \sqrt{(Z^{n+1})^2 + \rho^2}}{\varepsilon^2 - 1}$$

where

$$(9.7) \quad \rho(z) = \inf\{R > 0 : \mathcal{U} \subset B_R(z)\}.$$

Clearly $\bar{a}(Z)$ is the maximal value of larger semiaxis of hyperboloid H such that Γ_H is visible from \mathcal{U} in the e_{n+1} direction. In other words $H(x, \bar{a}(Z), Z)$ is the lowest possible hyperboloid with focus $Z \in \Sigma$ such that $H(x, \bar{a}(Z), Z) \geq 0$. Thus for $a \in (0, \bar{a}(Z)]$ we have $H(\cdot, a, Z) \in \mathbb{H}_0^+(\mathcal{U}, \mathcal{V})$. To check (9.6) we fix Z and pick x_0 such that $\rho(z) = |x_0 - z|$. Since the ratio of distances of x_0 from lower focus Z' and the plane $\Pi_d = \{X \in \mathbb{R}^{n+1} : X^{n+1} = Z^{n+1} - \bar{a}\varepsilon - \bar{a}/\varepsilon\}$ is ε , it follows that $|x_0 Z'| = \varepsilon(Z^{n+1} - \bar{a}\varepsilon - \bar{a}/\varepsilon)$. On the other hand $|x_0 Z| - |x_0 Z'| = 2\bar{a}$. Consequently, we find that $\sqrt{(Z^{n+1})^2 + \rho^2(z)} = 2\bar{a} + \varepsilon(Z^{n+1} - \bar{a}\varepsilon - \bar{a}/\varepsilon)$ which gives (9.6).

Next we define the maximal level $L_0 = \sup_{\mathcal{U} \times \mathcal{V}} H(x, \bar{a}(Z), Z)$. Since

$$\begin{aligned} \max_{x \in \mathcal{U}} H(x, \bar{a}(Z), Z) &= Z^{n+1} - \varepsilon \bar{a}(Z) - \bar{a}(Z) = \frac{\sqrt{(Z^{n+1})^2 + \rho^2(z)} - Z^{n+1}}{\varepsilon - 1} \\ &= \frac{\rho^2(z)}{(\varepsilon - 1)(\sqrt{(Z^{n+1})^2 + \rho^2(z)} + Z^{n+1})} \end{aligned}$$

it follows that

$$(9.8) \quad L_0 = \sup_{\mathcal{V}} \frac{\rho^2(z)}{(\varepsilon - 1)(\sqrt{(Z^{n+1})^2 + \rho^2(z)} + Z^{n+1})} \leq \frac{1}{\varepsilon - 1} \sup_{\mathcal{V}} \frac{\rho^2(z)}{2Z^{n+1}}.$$

Next, we bound $H(\cdot, a, Z)$ by below for $a > 0$ close to zero. By definition (8.3) we have that for this case $H(x, a, Z) \sim Z^{n+1} - \frac{\rho(z)}{\sqrt{\varepsilon^2 - 1}}$. We demand $Z^{n+1} - \frac{\rho(z)}{\sqrt{\varepsilon^2 - 1}} \geq 2L_0$ or equivalently in lieu of (9.8)

$$Z^{n+1} \geq \rho(z) \left[\frac{1}{\sqrt{\varepsilon^2 - 1}} + \frac{2\rho(z)}{(\varepsilon - 1)(\sqrt{(Z^{n+1})^2 + \rho^2(z)} + Z^{n+1})} \right].$$

But clearly $\frac{2\rho(z)}{(\varepsilon - 1)(\sqrt{(Z^{n+1})^2 + \rho^2(z)} + Z^{n+1})} \leq 2/(\varepsilon - 1)$. Therefore it is enough to assume that $Z^{n+1} \geq [\frac{2}{\varepsilon - 1} + \frac{1}{\sqrt{\varepsilon^2 - 1}}]\rho(z)$ which is exactly (2.11). It follows that if Σ satisfies (2.11) then $\tilde{\Sigma} = \Sigma + Me_{n+1}$, $M \gg 1$ also does.

Let $\mathbf{a} = (a_1, \dots, a_N)$, $a_i \in (0, \bar{a}(Z_i)]$, $i = 1, \dots, N$ and set

$$H(\mathbf{a}, x) = \min [H(x, a_1, Z_1), \dots, H(x, a_N, Z_N)].$$

We also let $\mathcal{E}_i(\mathbf{a}) = \{x \in \mathcal{U} : H(\mathbf{a}, x) = H(x, a_i, Z_i)\}$ be the i -th visibility sets and

$$\mathcal{A}^N = \left\{ \mathbf{a} \in \prod_{i=1}^N (0, \bar{a}_i(Z_i)] : \int_{\mathcal{E}_i(\mathbf{a})} f \leq C_i, \int_{\mathcal{E}_N(\mathbf{a})} f \geq C_N, \quad i = 1, \dots, N-1 \right\}.$$

From (2.11) it follows that \mathcal{A}^N is not empty for taking a_i , $1 \leq i \leq N-1$ close to zero and $a_N = \bar{a}_N(Z_N)$ one readily gets that such \mathbf{a} is in \mathcal{A}^N .

The visibility sets $\mathcal{E}_i(\mathbf{a})$ enjoy the following property: if for some $a_k < \bar{a}(Z_k)$ we set $\mathbf{a}_\delta^k = (a_1, \dots, a_k + \delta, \dots, a_N)$ and $\mathbf{a} = (a_1, \dots, a_N)$ for $\delta > 0$ small, then

$$(9.9) \quad \mathcal{E}_k(\mathbf{a}) \subset \mathcal{E}_k(\mathbf{a}_\delta^k) \quad \text{while} \quad \mathcal{E}_i(\mathbf{a}_\delta^k) \subset \mathcal{E}_i(\mathbf{a}), i \neq k.$$

This can be seen for $N = 2$ by simple geometric considerations, and general case is by induction.

Let $\mathbf{a} = \sup_{\mathbf{a} \in \mathcal{A}^N} \sum_{i=1}^N a_i$ and $\hat{\mathbf{a}} \in \mathcal{A}^N$ be such that the supremum is realised, i.e. $\mathbf{a} = \sum_{i=1}^N \hat{a}_i$. We claim that $H(\hat{\mathbf{a}}, x)$ solves the refractor problem with measure g_N . If not, then there is i_0 , say $i_0 = 1$, such that $\int_{\mathcal{E}_1(\hat{\mathbf{a}})} f < C_1$. Then in view of the energy balance condition this implies $\int_{\mathcal{E}_N(\hat{\mathbf{a}})} f > C_N$. For $\delta > 0$ small $F_N(\delta) = \int_{\mathcal{E}_N(\hat{\mathbf{a}}_\delta^1)} f(x) dx \geq C_N$ because $F_k(\delta)$ is continuous function of δ . Furthermore, using (9.9) it follows that $\mathbf{a}_\delta^1 \in \mathcal{A}^N$ which is a contradiction. Now the proof of Theorem C1 follows from the above polyhedral approximation $H(\hat{\mathbf{a}}, x)$ as $N \rightarrow \infty$ and the weak convergence of measures $\beta_{H,f}$, Lemma 9.1. \square

10. AN APPROXIMATION LEMMA

10.1. Refraction cone. Recall that for smooth refractors the unit direction of the refracted ray is

$$Y = \varepsilon \left(e_{n+1} + \gamma \left[\sqrt{(\gamma \cdot e_{n+1})^2 - \kappa} - \gamma \cdot e_{n+1} \right] \right),$$

see (4.4). This formula may be generalized for non smooth refractors as follows: let γ_1, γ_2 be the normals of two supporting planes of u at x . Then for any two constants c_1, c_2 the unit vector $\gamma_{c_1 c_2} = \frac{c_1 \gamma_1 + c_2 \gamma_2}{|c_1 \gamma_1 + c_2 \gamma_2|}$ generates a mapping to the unit sphere \mathbb{S}^{n+1} given by

$$\gamma_{c_1 c_2} \mapsto \varepsilon \left(e_{n+1} + \gamma_{c_1 c_2} \left[\sqrt{(\gamma_{c_1 c_2} \cdot e_{n+1})^2 - \kappa} - \gamma_{c_1 c_2} \cdot e_{n+1} \right] \right).$$

Definition 10.1. For $\gamma_1, \gamma_2 \in \mathbb{S}^{n+1}$ the refractor cone at $\xi \in \mathbb{R}^{n+1}$ is defined as

$$C_{\xi, \gamma_1, \gamma_2} = \left\{ Z \in \mathbb{R}^{n+1} : \frac{Z - \xi}{|Z - \xi|} = \varepsilon \left(e_{n+1} + \gamma_{c_1 c_2} \left[\sqrt{(\gamma_{c_1 c_2} \cdot e_{n+1})^2 - \kappa} - \gamma_{c_1 c_2} \cdot e_{n+1} \right] \right) \right\}.$$

One can easily verify that $C_{\xi, \gamma_1, \gamma_2}$ is a convex cone. Indeed, for any $\gamma_0 \perp \text{Span}\{\gamma_1, \gamma_2\}$ we have that $\frac{Z - \xi}{|Z - \xi|} \cdot \gamma_0 = \varepsilon(e_{n+1} \cdot \gamma_0)$. Thus $C_{\xi, \gamma_1, \gamma_2}$ is a cone.

In view of Lemma 8.1 $\|Du\|_{L^\infty(\mathcal{U})} \leq \frac{1}{\sqrt{\varepsilon^2 - 1}}$ for any admissible $u \in \overline{\mathbb{W}}_H(\mathcal{U}, \mathcal{V})$, and $\sqrt{(\gamma \cdot e_{n+1})^2 - \kappa}$ is well defined thanks to this gradient estimate.

10.2. Contact set. In this subsection we study the contact set of two hyperboloids

$$\Lambda = \{x \in \mathbb{R}^n : H(x, a_1, Z_1) = H(x, a_2, Z_2)\}$$

where $Z_i \in \Sigma, a_i > 0, i = 1, 2$. We show that Λ is a conic section. To check this we simplify the equation

$$Z_1^{n+1} - Z_2^{n+1} - \varepsilon(a_1 - a_2) - \sqrt{a_1^2 + \frac{|x - z_1|^2}{\varepsilon^2 - 1}} = -\sqrt{a_2^2 + \frac{|x - z_2|^2}{\varepsilon^2 - 1}}$$

by squaring both sides of it. Then denoting $C = Z_1^{n+1} - Z_2^{n+1} - \varepsilon(a_1 - a_2)$ we infer

$$C^2 - 2C\sqrt{a_1^2 + \frac{|x - z_1|^2}{\varepsilon^2 - 1}} + a_1^2 + \frac{|x - z_1|^2}{\varepsilon^2 - 1} = a_2^2 + \frac{|x - z_2|^2}{\varepsilon^2 - 1}.$$

Recognizing the terms and denoting $D = \frac{1}{2} \left[\frac{|z_1|^2 - |z_2|^2}{\varepsilon^2 - 1} + C^2 + a_1^2 - a_2^2 \right]$ we get

$$D + \frac{x \cdot (z_2 - z_1)}{\varepsilon^2 - 1} = C\sqrt{a_1^2 + \frac{|x - z_1|^2}{\varepsilon^2 - 1}}.$$

By choosing a suitable coordinate system we can assume that $z_2 - z_1$ is collinear to the unit direction of x_1 axis. Thus

$$D + \frac{x_1 |z_2 - z_1|}{\varepsilon^2 - 1} = C\sqrt{a_1^2 + \frac{|x - z_1|^2}{\varepsilon^2 - 1}}.$$

Finally squaring both sides of the last identity and assuming that $E := \frac{|z_1 - z_2|^2}{\varepsilon^2 - 1} - C^2 \neq 0$ we infer

$$\begin{aligned} C^2 \frac{|x' - z_1'|^2}{\varepsilon^2 - 1} &= D^2 - C^2 a_1^2 + \frac{1}{\varepsilon^2 - 1} \left(2Dx_1 |z_1 - z_2| + \frac{x_1^2 |z_1 - z_2|^2}{\varepsilon^2 - 1} - C^2 [x_1^2 - 2x_1 z_1^1 + (z_1^1)^2] \right) \\ &= D^2 - C^2 a_1^2 + \frac{E}{\varepsilon^2 - 1} \left(x_1^2 + 2x_1 \frac{D|z_1 - z_2| + C^2 z_1^1}{E} - \frac{C^2 (z_1^1)^2}{E} \right) \\ &= D^2 - C^2 a_1^2 + \frac{E}{\varepsilon^2 - 1} \left(\left[x_1 + \frac{D|z_1 - z_2| + C^2 z_1^1}{E} \right]^2 - \left[\frac{D|z_1 - z_2| + C^2 z_1^1}{E} \right]^2 - \frac{C^2 (z_1^1)^2}{E} \right) \\ &= F + \frac{E}{\varepsilon^2 - 1} \left[x_1 + \frac{D|z_1 - z_2| + C^2 z_1^1}{E} \right]^2 \end{aligned}$$

where

$$(10.1) \quad E = \frac{|z_1 - z_2|^2}{\varepsilon^2 - 1} - C^2, \quad F = D^2 - C^2 a_1^2 - \frac{E}{\varepsilon^2 - 1} \left(\left[\frac{D|z_1 - z_2| + C^2 z_1^1}{E} \right]^2 + \frac{C^2 (z_1^1)^2}{E} \right).$$

Note that if $E = 0$ then Λ is a paraboloid. Otherwise

$$\Lambda \text{ is } \begin{cases} \text{the sheet of hyperboloid } \frac{|x' - z_1'|^2}{a^2} = F + \frac{|x_1 - x_1^0|^2}{b^2} \text{ passing through } x^0, & \text{if } E > 0, \\ \text{the ellipsoid } \frac{|x' - z_1'|^2}{A^2} + \frac{|x_1 - x_1^0|^2}{B^2} = F \text{ with } \frac{A^2}{B^2} = \frac{C^2 - |z_1 - z_2|^2}{C^2}, & \text{if } E < 0. \end{cases}$$

We see that the rotational axis of Λ for both cases $E < 0$ and $E > 0$ is parallel to the direction of $z_2 - z_1$. Moreover, if Λ is an ellipsoid then this direction corresponds to the larger semiaxis. This observation will be used in the proof of Lemma 10.1 below.

10.3. R-convexity of \mathcal{V} .

Definition 10.2. We say that $\mathcal{V} \subset \Sigma$ is *R-convex* with respect to a point $\xi \in [0, \infty) \times \mathcal{U}$ if for any two unit vectors γ_1, γ_2 the intersection $C_{\xi, \gamma_1, \gamma_2} \cap \mathcal{V}$ is connected. If \mathcal{V} is R-convex with respect to any $\xi \in [0, \infty) \times \mathcal{U}$ then we simply say that \mathcal{V} is *R-convex*.

In particular a geodesic ball on the convex surface Σ is an example of R-convex \mathcal{V} .

10.4. Local supporting function is also global. In the Definition 8.1 of admissibility the supporting hyperboloid H is staying above u in whole \mathcal{U} . Consequently, one may wonder if the locally admissible functions (i.e. H stays above u only in a vicinity of the contact point) are still in $\overline{\mathbb{W}}_H(\mathcal{U}, \mathcal{V})$. This issue was addressed by G. Loeper in [17] for the optimal transfer problems. We have

Lemma 10.1. *Under the condition (2.8) a local supporting hyperboloid is also global.*

Proof. The proof is very similar to that of in [17], [25]. Let $H_i(x) = H(x, a_i, Z_i), i = 1, 2$ be two global supporting hyperboloids of u at x_0 such that the contact set $\Lambda \neq \{x_0\}$. Thus u is not differentiable at x_0 . To fix the ideas take $x_0 = 0$. If γ_i is the normal of the graph of $H_i, i = 1, 2$ at x_0 then for any $\theta \in (0, 1)$ there is $Z_\theta \in \Sigma \cap \mathcal{C}_{0, \gamma_1, \gamma_2}$ and $a_\theta > 0$ such that $H(x) = H(x, a_\theta, Z_\theta)$ is a local supporting hyperboloid of u at 0 and

$$(10.2) \quad DH_\theta(0) = \theta DH_1(0) + (1 - \theta)DH_2(0).$$

Observe that the correspondence $\theta \mapsto Z_\theta$ is one-to-one thanks to the assumption (2.4), see Remark 7.1. By choosing a suitable coordinate system we can assume that $DH_1(0) - DH_2(0) = (0, \dots, 0, \alpha)$ for some $\alpha > 0$. Then we have that for all $0 < \theta < 1$

$$(10.3) \quad \begin{aligned} \min[H_1(x), H_2(x)] &\leq \theta H_1(x) + (1 - \theta)H_2(x) \\ &= u(0) + [DH_2(0) + \alpha\theta]x_n + \frac{1}{2} [\theta D^2 H_1(0) + (1 - \theta)D^2 H_2(0)] x \otimes x + o(|x|^2) \end{aligned}$$

where the last line follows from Taylor's expansion.

Using the notations of Section 6 we have that

$$D^2 H_\theta(0) = - \frac{G(x_0, u(0), p_1 + \theta(p_2 - p_1))}{\varepsilon \kappa}$$

where we set $p_i = DH_i(0), i = 1, 2$ and used (10.2). For all unit vectors τ perpendicular to x_n axis we have

$$(10.4) \quad \begin{aligned} \frac{d^2}{d\theta^2} D_{\tau\tau}^2 H_\theta(0) &= - \frac{d^2}{d\theta^2} \frac{G^{ij}(0, u(0), p_1 + \theta(p_2 - p_1)) \tau_i \tau_j}{\varepsilon \kappa} \\ &= -\alpha^2 \frac{\partial^2}{\partial p_n^2} \frac{G^{jj}(0, u(0), p_1 + \theta(p_2 - p_1)) \tau_i \tau_j}{\varepsilon \kappa} \\ &\leq -\alpha^2 c_0 \end{aligned}$$

where the last line follows from (2.8) with $c_0 > 0$, see also (6.5) and Section 7.3.

Therefore

$$(10.5) \quad D_{\tau\tau}^2 H_\theta(0) \geq \theta D_{\tau\tau}^2 H_1(0) + (1 - \theta)D_{\tau\tau}^2 H_2(0) + \hat{c}_0 \theta(1 - \theta) |p_1 - p_2|^2$$

where \hat{c}_0 depends on c_0 .

Observe that at 0 we have

$$(10.6) \quad \theta p_1 + (1 - \theta)p_2 = \frac{1}{\varepsilon^2 - 1} \frac{z_\theta}{\sqrt{a_\theta^2 + \frac{|z_\theta|^2}{\varepsilon^2 - 1}}}, \quad \varphi(z_\theta) - u(0) - \varepsilon a_\theta = \sqrt{a_\theta^2 + \frac{|z_\theta|^2}{\varepsilon^2 - 1}}$$

where we assume that Σ is the graph of a function φ such that $\psi(Z) = Z^{n+1} - \varphi(z)$ satisfies (2.8). From these $n + 1$ equations we see that z_θ and a_θ are smooth functions of $\theta p_1 + (1 - \theta)p_2$. This yields the following crude estimate for the remaining second order derivatives

$$(10.7) \quad |\theta D_{jn}^2 H_1(0) + (1 - \theta)D_{jn}^2 H_2(0) - D_{jn}^2 H(0)| \leq C\theta(1 - \theta)|p_1 - p_2|^2, \quad j = 1, \dots, n$$

where C depends of C^2 form of φ . Consequently, after plugging (10.5) and (10.7) into (10.3) and recalling that $|p_1 - p_2| = \alpha$ we conclude

$$\begin{aligned}
\min[H_1(x), H_2(x)] &\leq u(0) + [p_2 + \alpha\theta]x_n + \frac{1}{2} [\theta D^2 H_1(0) + (1 - \theta) D^2 H_2(0)] x \otimes x + o(|x|^2) \\
&= u(0) + [p_2 + \alpha\theta]x_n + \frac{1}{2} D^2 H_\theta(0) x \otimes x - \widehat{c}_0 \theta (1 - \theta) \alpha^2 \sum_{j=1}^{n-1} x_j^2 + C \theta (1 - \theta) \alpha^2 |x_n| |x| + o(|x|^2) \\
&= u(0) + [p_2 + \alpha\theta]x_n + \frac{1}{2} D^2 H_\theta(0) x \otimes x - \\
&\quad - \widehat{c}_0 \theta (1 - \theta) \alpha^2 |x|^2 + \theta (1 - \theta) \alpha^2 (C |x_n| |x| + \widehat{c}_0 x_n^2) + o(|x|^2) \\
&= u(0) + [p_2 + \alpha\theta]x_n + \frac{1}{2} D^2 H_\theta(0) x \otimes x - \\
&\quad - \frac{\widehat{c}_0}{2} \theta (1 - \theta) \alpha^2 |x|^2 + \theta (1 - \theta) \alpha^2 \left(\frac{2C^2}{\widehat{c}_0} + \widehat{c}_0 \right) x_n^2 + o(|x|^2)
\end{aligned}$$

where the last line follows from Hölder's inequality. Now fixing θ_0 as in (10.2) and using the estimate $|D^2 H_\theta(0) - D^2 H_{\theta_0}(0)| \leq C|\theta - \theta_0|$ (with C depending on φ) we obtain

$$\begin{aligned}
\min[H_1(x), H_2(x)] &\leq u(0) + [p_2 + \alpha\theta_0]x_n + \frac{1}{2} D^2 H_{\theta_0}(0) x \otimes x + \\
&\quad + \alpha [\theta - \theta_0]x_n + \frac{1}{2} D^2 H_\theta(0) x \otimes x - \frac{1}{2} D^2 H_{\theta_0}(0) x \otimes x \\
&\quad - \frac{\widehat{c}_0}{2} \theta (1 - \theta) \alpha^2 |x|^2 + \theta (1 - \theta) \alpha^2 \left(\frac{2C^2}{\widehat{c}_0} + \widehat{c}_0 \right) x_n^2 + o(|x|^2) \\
&\leq H_{\theta_0}(x) + \alpha [\theta - \theta_0]x_n + C |\theta_0 - \theta| |x|^2 - \\
&\quad - \frac{\widehat{c}_0}{2} \theta (1 - \theta) \alpha^2 |x|^2 + \alpha^2 \left(\frac{2C^2}{\widehat{c}_0} + \widehat{c}_0 \right) x_n^2 + o(|x|^2).
\end{aligned}$$

Choosing $\theta = \theta_0 - x_n \alpha \left(\frac{2C^2}{\widehat{c}_0} + \widehat{c}_0 \right)$ such that $|x_n| < \delta$ with sufficiently small δ we finally obtain

$$\begin{aligned}
(10.8) \quad \min[H_1(x), H_2(x)] &\leq H_{\theta_0}(x) + C |\theta_0 - \theta| |x|^2 - \frac{\widehat{c}_0}{2} \theta (1 - \theta) \alpha^2 |x|^2 + o(|x|^2) = \\
&= H_{\theta_0}(x) - \frac{\widehat{c}_0}{2} \theta (1 - \theta) \alpha^2 |x|^2 + o(|x|^2) \\
&\leq H_{\theta_0}(x), \quad \forall x \in B_\delta.
\end{aligned}$$

This, in particular, implies that H_{θ_0} is a local supporting hyperboloid near $x = 0$.

It remains to check that H_{θ_0} is also a global supporting hyperboloid. The set $\Lambda_{1,2} = \{x \in \mathbb{R}^n : H_1(x) = H_2(x)\}$ passes through 0 and splits \mathcal{U} into two parts \mathcal{U}^+ and \mathcal{U}^- (recall that Λ_{12} is a conic section, see Section 10.2). It follows from (10.8) that the contact sets $\Lambda_{i,\theta_0} = \{x \in \mathbb{R}^n : H_i(x) = H_{\theta_0}(x)\}, i = 1, 2$ are tangent to Λ_{12} from one side in some vicinity of 0, say in \mathcal{U}^+ . If there is $\bar{x}_0 \neq 0$ such that, say, $\bar{x}_0 \in \Lambda_{1,2} \cap \Lambda_{1,\theta_0}$ then $H_1(\bar{x}_0) = H_2(\bar{x}_0) = H_{\theta_0}(\bar{x}_0)$ and $DH_{\theta_0}(\bar{x}_0) = \bar{\theta}_0 DH_1(\bar{x}_0) + (1 - \bar{\theta}_0) DH_2(\bar{x}_0)$ with possibly different $\bar{\theta}_0$. Observe that by construction the ray emitted from \bar{x}_0 in the direction of e_{n+1} after refraction from H_1, H_2 and H_{θ_0} hits the point Z_1, Z_2 and Z_{θ_0} , respectively. Then repeating the argument above with 0 replaced by \bar{x}_0 and θ_0 by $\bar{\theta}_0$ (but keeping H_{θ_0} fixed), we can see that (10.8) is satisfied in $B_\delta(\bar{x}_0)$ implying that Λ_{1,θ_0} is tangent with $\Lambda_{1,2}$ at \bar{x}_0 and lies in \mathcal{U}^+ . Thus H_{θ_0} is a global supporting hyperboloid. \square

As an application of Lemma 10.1 we have the following approximation result.

Lemma 10.2. *If $u \in \overline{\mathbb{W}}_H(B_r, \Sigma)$ then*

- (i) $u_\varepsilon(x) + K(r^2 - |x|^2) \in \overline{\mathbb{W}}_H(B_r, \tilde{\Sigma})$ where u_ε is the standard mollification of u , $K > 0$ and $\tilde{\Sigma} = \Sigma + M e_{n+1}$ for some large constant $M > 0$,

(ii) $u_\varepsilon(x) + K(r^2 - |x|^2)$ is a classical subsolution of (6.2).

Proof. (i) It is well known that u_ε is concave and $\|Du_\varepsilon\|_{L^\infty(B_r)} \leq \|Du\|_{L^\infty(B_r)} < \frac{1}{\sqrt{\varepsilon^2 - 1}}$. Therefore if K is fixed then we can choose r so small that

$$(10.9) \quad \|D\bar{u}_\varepsilon\|_{L^\infty(B_r)} \leq \|Du\|_{L^\infty(B_r)} + 2Kr < \frac{1}{\sqrt{\varepsilon^2 - 1}}.$$

Moreover $K(r^2 - |x|^2)$ is concave, hence $\bar{u}_\varepsilon = u_\varepsilon(x) + K(r^2 - |x|^2)$ is concave as well. Notice that $D^2\bar{u}_\varepsilon = D^2u_\varepsilon - 2K\text{Id} \leq -2K\text{Id} < 0$ implying that \bar{u}_ε is strictly concave. In order to bound the curvature of $\Gamma_{\bar{u}_\varepsilon}$ from below we recall that for fixed Z , $H(\cdot, a, Z)$ becomes flatter as $a \rightarrow \infty$ because

$$D^2H = -\frac{1}{(\varepsilon^2 - 1)\sqrt{a^2 + \frac{|x-z|^2}{\varepsilon^2 - 1}}} \left[\text{Id} - \frac{(x-z) \otimes (x-z)}{(\varepsilon^2 - 1)a^2 + |x-z|^2} \right].$$

In particular, for large K and a we will have $-D^2\bar{u}_\varepsilon \geq 2K\text{Id} \geq -D^2H$. Consequently, for each $x \in \mathcal{U}$ there is Z and $a > 0$ such that $H(\cdot, a, Z)$ touches \bar{u}_ε from above at x , in some neighbourhood of x . Furthermore, from Lemma 8.3 on confocal expansion we can choose a, \tilde{Z} so that $\tilde{Z} \in \tilde{\Sigma} = \Sigma + Me_{n+1}$, $M \gg 1$. Finally applying Lemma 10.1 we infer that $H(\cdot, a, \tilde{Z})$ is a global supporting hyperboloid of u at x and thus $\bar{u}_\varepsilon \in \overline{\mathbb{W}}_H(\mathcal{U}, \tilde{\Sigma})$.

(ii) By direct computation we have

$$(10.10) \quad \begin{aligned} \mathcal{M} &= -D^2\bar{u}_\varepsilon - \frac{G(x, \bar{u}_\varepsilon, D\bar{u}_\varepsilon)}{\varepsilon\kappa} = -D^2u_\varepsilon + 2K\text{Id} - \frac{G(x, \bar{u}_\varepsilon, D\bar{u}_\varepsilon)}{\varepsilon\kappa} \geq \\ &\geq 2K\text{Id} - \frac{G(x, \bar{u}_\varepsilon, D\bar{u}_\varepsilon)}{\varepsilon\kappa}. \end{aligned}$$

By definition, (6.1) we have

$$\frac{G(x, \bar{u}_\varepsilon, D\bar{u}_\varepsilon)}{\varepsilon\kappa} = \frac{[q+1](\text{Id} - \varepsilon^2\kappa D\bar{u}_\varepsilon \otimes D\bar{u}_\varepsilon)}{\varepsilon\kappa t(x, \bar{u}_\varepsilon, D\bar{u}_\varepsilon)} \leq \frac{C}{t(x, \bar{u}_\varepsilon, D\bar{u}_\varepsilon)}$$

with some tame constant $C > 0$ depending only on ε . Recall that by (2.10) $t = \frac{(M+[Z^{n+1}-\bar{u}_\varepsilon])}{Y^{n+1}} \sim \frac{M}{c(\varepsilon)}$, $Z \in \Sigma$. Therefore choosing M large enough, one sees that $\mathcal{M} \geq \left[2K - \frac{Cc(\varepsilon)}{M}\right]\text{Id} \geq K\text{Id}$ if $K > \frac{Cc(\varepsilon)}{M}$. Fixing $K \geq \max\left[\frac{Cc(\varepsilon)}{M}, \sup|h|^\frac{1}{n}\right]$, where h is defined by (6.4) and choosing r small enough such that (10.9) holds we finally arrive at $\det\left[-D^2\bar{u}_\varepsilon - \frac{G(x, \bar{u}_\varepsilon, D\bar{u}_\varepsilon)}{\varepsilon\kappa}\right] \geq |h|$ and the proof is complete. \square

11. A-TYPE WEAK SOLUTIONS AND THE LEGENDRE-LIKE TRANSFORM

In this section we are concerned with the second notion of weak solution to **(RP)**. For $u \in \overline{\mathbb{W}}_H(\mathcal{U}, \mathcal{V})$ let us consider the mapping $\mathcal{R}_u : \mathcal{U} \rightarrow \Sigma$ defined as

$$\mathcal{R}_u(x) = \{Z \in \Sigma : \text{there is a supporting hyperboloid } H(\cdot, a, Z) \text{ of } u \text{ at } x\}.$$

Let $E \subset \mathcal{U}$ be a Borel set and put

$$\mathcal{R}_u(E) = \bigcup_{x \in E} \mathcal{R}_u(x).$$

Our primary goal is to prove that $\mathcal{R}_u(E)$ is measurable with respect to the restriction of \mathcal{H}^n on Σ for any Borel set $E \subset \mathcal{U}$. That done, we can proceed as in [11] and establish that the set function $\alpha_{u,g}$ is σ -additive measure.

To take advantage of the geometric intuition coming from supporting hyperboloids of $u \in \overline{\mathbb{W}}_H(\mathcal{U}, \mathcal{V})$ it is convenient to define the Legendre-like transformation of u . We use the construction of smallest focal parameter introduced by Xu-Jia Wang in [27] (equation (1.15)). Let $u \in \overline{\mathbb{W}}_H(\mathcal{U}, \Sigma)$ and $Z \in \Sigma$ be a fixed point. Then the smallest semi-axis among all hyperboloids $H(\cdot, a, Z)$ that stay above u is

$$a_0 = \sup_{a \in I(Z)} a, \quad I(Z) = \{a > 0 : H(x, a, Z) \geq u(x) \text{ in } \mathcal{U}\}.$$

Suppose that $H(\cdot, a_0, Z)$ touches u at $x_0 \in \mathcal{U}$ then

$$u(x_0) = \psi(z) - a_0\varepsilon - \sqrt{a_0^2 + \frac{(x_0 - z)^2}{\varepsilon^2 - 1}}.$$

From here we can easily find that

$$(11.1) \quad a_0 = \frac{1}{\varepsilon^2 - 1} \left\{ \varepsilon[u(x_0) - \psi(z)] - \sqrt{[u(x_0) - \psi(z)]^2 + (x_0 - z)^2} \right\}.$$

Alternatively, one can use the property that the distance of a point P on hyperboloid from lower focus Z' is ε times the distance of P from the hyperplane $\Pi_d = \{X \in \mathbb{R}^{n+1} : X^{n+1} = Z^{n+1} - a\varepsilon - \frac{a}{\varepsilon}\}$ (which in one dimensional case is the directrix). Since by definition of hyperboloid $|PZ| - |PZ'| = 2a$ and $|PZ'| = \varepsilon \text{dist}(P, \Pi_d)$ we infer $|PZ| = 2a + \varepsilon([\psi(z) - u(x_0)] - a\varepsilon - \frac{a}{\varepsilon}) = -a(\varepsilon^2 - 1) + \varepsilon([\psi(z) - u(x_0)])$ and (11.1) follows.

11.1. A-type weak solutions.

Definition 11.1. Let $u \in \overline{\mathbb{W}}_H(\mathcal{U}, \Sigma)$ then

$$(11.2) \quad v(z) = \inf_{x \in \mathcal{U}} \left\{ \varepsilon[\psi(z) - u(x)] - \sqrt{[u(x) - \psi(z)]^2 + (x - z)^2} \right\}$$

is called the Legendre-like transformation of u .

If $\text{dist}(\mathcal{U}, \mathcal{V}) > 0$ and $\psi \in C^2$ then the function $\mathcal{L}_x(z) = \varepsilon[\psi(z) - u(x)] - \sqrt{[u(x) - \psi(z)]^2 + (x - z)^2}$ is C^2 -smooth for any fixed $x \in \mathcal{U}$. Since v is the upper envelope of C^2 smooth functions $\mathcal{L}_x, x \in \mathcal{U}$ (x being the parameter) then $v(z)$ is semi-convex. Next lemma gives an important characterization of $v(z)$.

Lemma 11.1. Let v be the Legendre-like transformation of $u \in \overline{\mathbb{W}}_H(\mathcal{U}, \Sigma)$. Then

- (i) $v(z) = \varepsilon[\psi(z) - u(x_0)] - \delta_u(x_0, z)$ if $Z = (z, \psi(z)) \in \mathcal{R}_u(x_0)$ where $\delta_u(x, z) = \sqrt{[u(x) - \psi(z)]^2 + (x - z)^2}$,
- (ii) $v(z)$ is semi-concave.

Proof. By definition $v(z)$ is locally bounded, non-negative, lower semi-continuous function. Let $\delta_u(x, z)$ denote the distance between the points of graph Γ_u and Σ . To check (i) we first observe that by definition of $v(z)$, see (11.2), we have $v(z) \leq \varepsilon[\psi(z) - u(x_0)] - \delta_u(x_0, z)$. If $v(z) < \varepsilon[\psi(z) - u(x_0)] - \delta_u(x_0, z)$ it follows from (11.1) and the discussion above that $H(\cdot, a_0, Z)$ is a supporting hyperboloid of u at x_0 , where $a_0 = (\varepsilon^2 - 1)^{-1}(\varepsilon[\psi(z) - u(x_0)] - \delta_u(x_0, z))$ because $Z \in \mathcal{R}_u(x_0) \subset \Sigma$. On the other hand, there is a sequence $\{x_k\}$ in \mathcal{U} such that $x_k \rightarrow \bar{x}_0 \in \overline{\mathcal{U}}$ and $\lim_{x_k \rightarrow \bar{x}_0} (\varepsilon[\psi(z) - u(x_k)] - \delta_u(x_k, z)) = v(z)$. Setting $\bar{a}_0 = (\varepsilon^2 - 1)^{-1}v(z)$ we conclude that $H(\cdot, \bar{a}_0, Z)$ is touching Γ_u at \bar{x}_0 . By construction $\bar{a}_0 < a_0$ and it follows from confocal expansion of hyperboloids 8.4 that $H(\cdot, \bar{a}_0, Z) > H(\cdot, a_0, Z)$ in \mathcal{U} . But this inequality is in contradiction with the fact that $H(\cdot, a_0, Z)$ is a supporting hyperboloid of u at x_0 and $H(\cdot, \bar{a}_0, Z)$ touches Γ_u at \bar{x}_0 whilst staying above Γ_u .

To prove (ii) we let $\mathcal{L}_{x_0}(y) = \varepsilon[\psi(z) - u(x_0)] - \delta_u(x_0, y)$. Then

$$v(y) = \inf_{x \in \mathcal{U}} \left\{ \varepsilon[\psi(z) - u(x)] - \delta_u(x, y) \right\} \leq \varepsilon[\psi(z) - u(x_0)] - \delta_u(x_0, y)$$

which implies that $v(y) \leq \mathcal{L}_{x_0}(y)$ and $v(z) = \mathcal{L}_{x_0}(z)$, where $Z \in \mathcal{R}_u(x_0)$. We can regard $\mathcal{L}_{x_0}(y)$ as an upper supporting function of v at z . Differentiating \mathcal{L}_{x_0} twice in z variable we see that $|D^2 \mathcal{L}_{x_0}(z)| \leq \frac{C}{(\text{dist}(\mathcal{U}, \mathcal{V}))^3}$ for some tame constant $C > 0$, consequently $v(z) - C|z|^2$ is concave for large $C > 0$. \square

The main result of this section is contained in the following

Lemma 11.2. Let $\mathcal{S} = \{Z \in \mathcal{V} : \text{such that } Z \in \mathcal{R}_u(x_1) \cap \mathcal{R}_u(x_2), x_1 \neq x_2\}$. Then \mathcal{S} has vanishing surface measure on Σ .

Proof. Let us show that if $Z \in \mathcal{S}$ then the Legendre-like transformation of u is not differentiable at Z . This will suffice to conclude the proof because by definition v is semiconcave and hence by Aleksandrov's theorem twice differentiable almost everywhere. Let v be the Legendre-like transformation of u , then by Lemma 11.1 for any $Z \in \mathcal{R}_u(x_0)$ at which $v(z)$ is differentiable there holds

$$(11.3) \quad Dv(z) = \varepsilon D\psi(z) - (y(x) + D\psi(z)y^{n+1}(x)).$$

Indeed, $Dv(z) = \varepsilon D\psi(z) - \delta_u(x, z)^{-1} [(z - x) + D\psi(z)(\psi(z) - u(x))]$. From the definition of stretch function t it follows that $(z - x, \psi(z) - u(x)) = Y\delta_u(x, z)$ where $Y = (y, y^{n+1})$ is the unit direction of the refracted ray and (11.3) follows. Consequently, if $x_1 \neq x_2$ such that $\mathcal{R}_u(x_1) \cap \mathcal{R}_u(x_2) \ni Z$ then we must have

$$Dv(z) = -\frac{z - x_i + D\psi(z)(\psi(z) - x_i)}{\delta_u(x_i, z)} + \varepsilon D\psi(z), \quad i = 1, 2.$$

Equating the right hand sides for $i = 1$ and $i = 2$ we obtain

$$\frac{z - x_1 + D\psi(z)(\psi(z) - x_1)}{\delta_u(x_1, z)} = \frac{z - x_2 + D\psi(z)(\psi(z) - x_2)}{\delta_u(x_2, z)}$$

With the aid of this observation and (11.3) we can rewrite the last line as follows

$$y_1 + D\psi(z)y_1^{n+1} = y_2 + D\psi(z)y_2^{n+1} \text{ in } \mathbb{R}^n \quad \Rightarrow \quad Y_1 + (D\psi(z), -1)y_1^{n+1} = Y_2 + (D\psi(z), -1)y_2^{n+1}, \text{ in } \mathbb{R}^{n+1}.$$

The last identity implies that $Y_1 - Y_2$ is collinear to the normal of Σ at Z . Consequently, from the assumption (2.4) (see also (2.10)) we obtain that this is possible if and only if $Y_1 = Y_2$. Next, from $Y_1 = Y_2$ we have $y_1 = y_2$ and consequently we conclude that

$$(11.4) \quad \frac{z - x_1}{\delta_u(x_1, z)} = \frac{z - x_2}{\delta_u(x_2, z)}.$$

Taking the reciprocal of both sides in the last identity and recalling the definition of the distance $\delta_u(x, z)$ one gets

$$\frac{u(x_1) - \psi(z)}{|x_1 - z|} = \frac{u(x_2) - \psi(z)}{|x_2 - z|}$$

yielding

$$\begin{aligned} u(x_1) &= \psi(z) + \frac{|z - x_1|}{|z - x_2|} (u(x_2) - \psi(z)) \\ &= \psi(z) + \frac{\delta_u(x_1, z)}{\delta_u(x_2, z)} (u(x_2) - \psi(z)). \end{aligned}$$

On the other hand $Y_1^{n+1} = Y_2^{n+1}$ gives $u(x_1) - u(x_2) = \delta_u(x_2, z) - \delta_u(x_1, z)$ and hence combining this with the last equation yields

$$\psi(z) \left[1 - \frac{\delta_u(x_1, z)}{\delta_u(x_2, z)} \right] - u(x_2) \left[1 - \frac{\delta_u(x_1, z)}{\delta_u(x_2, z)} \right] = \delta_u(x_1, z) - \delta_u(x_2, z).$$

If $\delta_u(x_2, z) \neq \delta_u(x_1, z)$ then the last equality implies $u(x_2) - \psi(z) = \sqrt{(u(x_2) - \psi(z))^2 + (z - x_2)^2}$. Hence $x_2 = z$ and by (11.4) $x_1 = x_2$, which is contradiction. Thus we must have $\delta_u(x_2, z) = \delta_u(x_1, z)$ and in view of (11.4) this implies that $x_1 = x_2$, again contradicting our supposition. Therefore we infer that $v(z)$ cannot be differentiable at z . By Rademacher's theorem $v(z)$ is differentiable a.e. in z . Thus \mathcal{S} has vanishing surface measure. \square

Corollary 11.1. *For any $u \in \overline{\mathbb{W}}_H(\mathcal{U}, \mathcal{V})$ and any Borel subset $E \subset \mathcal{U}$ the set function*

$$(11.5) \quad \alpha_{u,g}(E) = \int_{\mathcal{R}_u(E)} g d\mathcal{H}^n$$

is a Radon measure.

Proof. In order to show that $\alpha_{u,g}$ is Radon measure it suffices to check that $\widetilde{\mathcal{F}} = \{E \subset \mathcal{U} : \mathcal{R}_u(E) \text{ is measurable}\}$ is a σ -algebra. This can be done exactly in the same way as in the proof of Proposition 9.1 c). It remains to recall that by Lemma 8.2, $\widetilde{\mathcal{F}}$ contains the closed sets. \square

Definition 11.2. A function $u \in \overline{\mathbb{W}}_H(\mathcal{U}, \mathcal{V})$ is said to be *A-type weak solution of (RP)* if $\int_E f(x)dx = \alpha_{u,g}(E)$ or any Borel set $E \subset \mathcal{U}$ and

$$(11.6) \quad \overline{\mathcal{V}} \subset \overline{\mathcal{R}_u(\mathcal{U})}, \quad |\{x \in \mathcal{U} : \mathcal{R}_u(x) \not\subset \mathcal{V}\}| = 0$$

This definition is natural, stating that the target domain \mathcal{V} is covered by the refracted rays and the endpoints of those rays that after refraction do not strike \mathcal{V} constitute a null set on \mathcal{U} . We shall establish the existence of A-type weak solution in the next section.

In closing this section we state the weak convergence result for the α -measures, see Corollary 11.1.

Lemma 11.3. Let u_k be a sequence of A-type weak solutions and α_k is the corresponding measure, defined by (11.5). If $u_k \rightarrow u$ uniformly on compact subsets of \mathcal{U} then u is R-concave and α_k weakly converges to $\alpha_{u,g}$.

The proof is very similar to that of Lemma 9.1 (modulo minor adjustments) and hence omitted.

Remark 11.1. The Legendre like transformation (11.2) can be used to reformulate (RP) as a nonlinear optimization problem, following the method set out in [15].

12. COMPARING A AND B TYPE WEAK SOLUTIONS: PROOF OF THEOREM C3-4

In this section we prove the equivalence of A and B type weak solutions under some conditions. These results are known for in \mathbb{R}^n for the sub-differential [4], [26]. Let $\varphi : \mathbb{R}^N \rightarrow \mathbb{R}^n$ be a Borel mapping and $\mu(\mathbb{R}^N) = \nu(\mathbb{R}^n) < \infty$ with μ, ν being two Radon measure on \mathbb{R}^N and \mathbb{R}^n , respectively. Then φ induces a (push-forward) measure on \mathbb{R}^n defined by $\varphi_{\#}\mu(E) = \mu(\varphi^{-1}(E))$ for Borel subsets $E \subset \mathbb{R}^n$. We say that a Borel mapping φ *measure preserving* if

$$(12.1) \quad \varphi_{\#}\mu(E) = \nu(E) \quad \text{for any Borel set } E \subset \mathbb{R}^n.$$

By the change of variables formula (12.1) can be rewritten in the following equivalent form

$$(12.2) \quad \int h(\varphi(x))d\mu = \int h(y)d\nu, \quad \forall h \in C(\mathbb{R}^n),$$

see [3].

Remark 12.1. If $u \in \mathbb{W}^+(\mathcal{U}, \Sigma)$ is the B-type solution of (RP), the existence of which is established in Section 9, then taking $\varphi(Z) = \mathcal{S}_u(Z)$, $N = n+1$, $d\mu = g d\mathcal{H}^n$, and ν being the Lebesgue measure one immediately observes that \mathcal{S}_u is measure preserving in the sense of (12.1) or (12.2).

Lemma 12.1. If $\mathcal{R}_u(x) \subset \mathcal{V}$ for a.e. $x \in \mathcal{U}$ then $\mathcal{R}_u(E) \subset \text{Hull}(\mathcal{V})$, where $\text{Hull}(\mathcal{V})$ is the R-convex hull of \mathcal{V} defined as the smallest R-convex subset of Σ containing \mathcal{V} .

Proof. We only have to consider the points where u is non-differentiable. Let u be non-differentiable at $x_0 \in \mathcal{U}$ and suppose that γ_1, γ_2 are the normals of two supporting planes of u at x_0 . The ray with endpoint x_0 after reflection will lie in the reflector cone $\mathcal{C}_{\xi_0, \gamma_1, \gamma_2}$, with $\xi_0 = (x_0, u(x_0))$ and the reflected ray will strike $\text{Hull}(\mathcal{V})$, because $\mathcal{C}_{\xi_0, \gamma_1, \gamma_2} \cap \text{Hull}(\mathcal{V})$ is connected. Considering all normals of supporting planes at x_0 we obtain the desired result. \square

Proposition 12.1. *Let Σ be R -convex with respect to $Q_m = \mathcal{U} \times (0, m)$, $m > 0$ and the densities f, g are positive. Then B -type weak solution is also of A -type.*

Proof. We split the proof into three parts.

1) First we show that for any compact $K_1 \subset \mathcal{U}$ there holds $\int_{K_2} g d\mathcal{H}^n \geq \int_{K_1} f(x) dx$ with $K_2 = \mathcal{R}_u(K_1)$. In other words the B -type solution is A -type subsolution. It is worthwhile to point out that for the proof of this inequality we don't need \mathcal{V} to be R -convex. Take $\eta \in C(\Sigma)$ such that $\eta \equiv 1$ on $K_2 \subset \Sigma$ and $0 \leq \eta \leq 1$. From (12.2) we see that

$$\int_{\mathcal{V}} \eta g d\mathcal{H}^n = \int_{\mathcal{U}} \eta(\mathcal{R}_u(x)) f(x) dx \geq \int_{K_1} f(x) dx.$$

Letting η to decrease to the characteristic function of K_2 , $h \downarrow \chi_{K_2}$ we infer

$$(12.3) \quad \int_{K_2} g d\mathcal{H}^n \geq \int_{K_1} f(x) dx.$$

Notice by Corollary 11.1 the measure $\alpha_{u,g}$ is Borel regular, therefore in the last inequality K_1 can be replaced by any Borel subset of \mathcal{U} . As a result we conclude from (12.3) that

$$(12.4) \quad \text{if } \mathcal{H}^n(\mathcal{R}_u(E)) = 0 \text{ then } |E| = 0.$$

2) Next, we prove the converse estimate of (12.3). Here we will utilize the R -convexity of \mathcal{V} . Take any compact $K_1 \in \mathcal{U}$ and apply Lemma 11.2 to conclude $\mathcal{H}^n(\mathcal{R}_u(K_1) \cap \mathcal{R}_u(\mathcal{U} \setminus K_1)) = 0$. Let us show that

$$(12.5) \quad |\mathcal{R}_u^{-1}(\mathcal{R}_u(K_1)) \setminus K_1| = 0$$

where $\mathcal{R}_u^{-1}(\mathcal{R}_u(K_1))$ is the pre-image of $\mathcal{R}_u(K_1)$. Denote $E = \mathcal{R}_u^{-1}(\mathcal{R}_u(K_1))$ and $G = K_1$. If $\mathcal{H}^n(E \setminus G) = 0$ then in view of (12.4) we obtain (12.5). Indeed, from the identity (9.2) it follows that

$$(12.6) \quad \begin{aligned} |\mathcal{R}_u(E \setminus G)| &= \left| [\mathcal{R}_u(E) \setminus \mathcal{R}_u(G)] \cup [\mathcal{R}_u(E \setminus G) \cap \mathcal{R}_u(G)] \right| \\ &= |\mathcal{R}_u(E \setminus G) \cap \mathcal{R}_u(G)| \\ &= 0 \end{aligned}$$

where to get the last line we used the definitions of E and G in order to obtain $\mathcal{R}_u(E) \setminus \mathcal{R}_u(G) = \mathcal{R}_u(K_1) \setminus \mathcal{R}_u(K_1) = \emptyset$ and Lemma 11.2. Thus (12.4) implies $0 = |E \setminus G| = |\mathcal{R}_u^{-1}(\mathcal{R}_u(K_1)) \setminus K_1|$.

Let $h \in C(\Sigma)$ such that $0 \leq h \leq 1$ and $h \geq \chi_{\mathcal{R}_u(K_1)}$. Since \mathcal{V} is R -convex it follows that $\mathcal{R}_u(K_1) \subset \text{Hull}\mathcal{V}$, see Lemma 12.1. If u is a B -type weak solution then (12.2) holds, see Remark 12.1. Therefore

$$\begin{aligned} \int_{\mathcal{U}} \eta(\mathcal{R}_u(x)) f(x) dx &= \int_{\mathcal{V}} \eta g d\mathcal{H}^n \\ &= \int_{\text{Hull}(\mathcal{V})} \eta g d\mathcal{H}^n \\ &\geq \int_{\mathcal{R}_u(K_1)} g d\mathcal{H}^n. \end{aligned}$$

Letting $\eta \rightarrow 0$ on compact subsets of $\mathcal{V} \setminus \mathcal{R}_u(K_1)$, it follows that $\eta(\mathcal{R}_u(x))$ uniformly converges to zero one the compact subsets of $\mathcal{U} \setminus \mathcal{R}_u^{-1}(\mathcal{R}_u(K_1))$. Consequently

$$\int_{\mathcal{R}_u(K_1)} g d\mathcal{H}^n \leq \int_{\mathcal{U}} \eta(\mathcal{R}_u(x)) f(x) dx \longrightarrow \int_{\mathcal{R}_u^{-1}(\mathcal{R}_u(K_1))} f(x) dx = \int_{K_1} f(x) dx$$

where the last line follows from (12.5). This implies that u is a supersolution.

3) It remains to check that u verifies the boundary condition (11.6). Suppose that there is $Z_0 \in \overline{\mathcal{V}}$ such that $Z_0 \notin \overline{\mathcal{R}_u(\mathcal{U})}$. Since u is of B -type, it follows that $\mathcal{S}_u(\mathcal{V}) = \mathcal{U}$ implying $x_0 \in \mathcal{S}_u(Z_0)$ in other words, there is a supporting hyperboloid $H(x, a_0, Z_0)$ at x_0 . Thus $Z_0 \in \mathcal{R}_u(x_0)$ which yields $\overline{\mathcal{V}} \subset \overline{\mathcal{R}_u(\mathcal{U})}$. From energy balance condition we have

$$\int_{\overline{\mathcal{R}_u(\mathcal{U})}} g d\mathcal{H}^n = \int_{\mathcal{U}} f(x) dx = \int_{\mathcal{V}} g d\mathcal{H}^n \Rightarrow \int_{\overline{\mathcal{R}_u(\mathcal{U})} \setminus \mathcal{V}} g = 0.$$

This yields $|\{x \in \mathcal{U} : \mathcal{R}_u(x) \not\subset \mathcal{V}\}| = 0$ for $f, g > 0$. \square

Remark 12.2. *We always have $\overline{\mathcal{V}} \subset \mathcal{R}_u(\mathcal{U})$, however if in addition Σ is R -convex then it follows that $\mathcal{R}_u(\mathcal{U}) \subset \mathcal{V}$. Thus we get the equality $\overline{\mathcal{R}_u(\mathcal{U})} = \overline{\mathcal{V}}$ for R -convex \mathcal{V} .*

12.1. Existence of A -type weak solutions: Proof of Theorem C4. Suppose that $\mathcal{V} \subset \Sigma$ and let $\text{Hull}(\mathcal{V})$ be the R -convex hull of \mathcal{V} . For small $\delta, \delta' > 0$ we consider

$$(12.7) \quad g_\delta(Z) = \begin{cases} g(Z) - \delta & \text{if } Z \in \mathcal{V} \\ \delta' & \text{if } Z \in \text{Hull}(\mathcal{V}) \setminus \mathcal{V} \end{cases}$$

where we choose δ, δ' so that g_δ satisfies the energy balance condition (1.3). By Proposition 9.2 for each g_δ there is a B -type weak solution which according to Proposition 12.1 is also of A -type. Moreover, from Remark 12.2 we infer

$$(12.8) \quad \overline{\mathcal{R}_{u_\delta}(\mathcal{U})} = \overline{\mathcal{V}}.$$

Sending $\delta \rightarrow 0$ we obtain from Lemma 11.3 that $u_\delta \rightarrow u$ and u is an A -type solution, i.e. (11.5) is satisfied, and

$$(12.9) \quad \overline{\mathcal{V}} \subset \overline{\mathcal{R}_u(\mathcal{U})}.$$

Since u is second order differentiable a.e. in \mathcal{U} it follows that \mathcal{R}_u is defined for a.e. $x \in \mathcal{U}$. Finally we want to show that $|S| = 0$ where $S = \{x \in \mathcal{U} : \exists Z \in \mathcal{R}_u(x) \text{ such that } Z \in \mathcal{R}_u(\mathcal{U}) \setminus \mathcal{V}\}$. Indeed, from energy balance condition (1.3) we have

$$\begin{aligned} \int_S f(x) dx &= \int_{\mathcal{U}} f(x) dx - \int_{\mathcal{U} \setminus S} f(x) dx = \\ &= \int_{\mathcal{U}} f(x) dx - \int_{\mathcal{V}} g d\mathcal{H}^n = 0. \end{aligned}$$

Since $f > 0$ we conclude that $|S| = 0$ and hence (11.6) holds and u is a weak A -type weak solution of **(RP)**. \square

Remark 12.3. *As the proof of Proposition 12.1 exhibited if \mathcal{V} is R -convex then $S = \emptyset$. If $S \neq \emptyset$ then u is only Lipschitz continuous. Therefore if \mathcal{V} is not R -convex then u may not be C^1 smooth, see Introduction. It is worthwhile to point out that even if $S = \emptyset$ then u may not be C^1 , and hence further assumptions must be imposed to assure the smoothness of u .*

13. DIRICHLET'S PROBLEM

This section concerns the Dirichlet problem for A -type weak solutions. We formally rewrite the equation (6.2) below

$$(13.1) \quad \mathcal{F}[u](x) = \frac{f(x)}{g \circ \mathcal{R}_u(x)}, \quad x \in \mathcal{U},$$

where for $u \in C^2(\mathcal{U})$, $\mathcal{F}[u](x)$ is the determinant of the Jacobian matrix of \mathcal{R}_u . For non-smooth solutions we give the following definition.

Definition 13.1. A function $u \in \overline{\mathbb{W}}_H(\mathcal{U}, \Sigma)$ is said to be a weak A -subsolution of (13.1) if for any Borel set E

$$(13.2) \quad \int_{\mathcal{R}_u(E)} g d\mathcal{H}^n \geq \int_E f(x) dx.$$

If $\alpha_{u,g}(E) = \int_E f(x) dx$ then we say that u is a weak A -solution. The class of all generalized A -subsolutions is denoted by $\mathcal{AS}^+(\mathcal{U})$.

For smooth and bounded $D \subset \Sigma$ and smooth function $\varphi : \overline{D} \rightarrow \mathbb{R}$ let us consider the Dirichlet problem

$$(13.3) \quad \begin{cases} \mathcal{F}[u](x) = \frac{f(x)}{g \circ \mathcal{R}_u(x)} & x \in D, \\ u = \varphi & x \in \partial D. \end{cases}$$

Our main objective here is to prove the existence and uniqueness of A -type weak solution to (13.3) for a smooth boundary data. In fact, for our purposes it suffices to consider the case where D is a ball of small radius. At this point we first establish the following comparison principle.

Proposition 13.1. Let u_i be weak solutions of (13.1) in \mathcal{U} with $f = f_i, i = 1, 2$, where $\Omega \subset \Pi$ is a smooth, bounded domain and conditions in Theorem C hold. Suppose that $\mathcal{R}_{u_1}(\Omega) \subset \overline{\Sigma}$, $f_1 < f_2$ in Ω and $u_1 \leq u_2$ on $\partial\Omega$. If Γ_1 , the graph of u_1 , lies in the region \mathcal{D} then we have $u_1 \leq u_2$ in Ω .

Proof. Suppose that $\Omega_1 = \{x \in \Omega : u_1(x) > u_2(x)\}$ is not empty. Let $x_0 \in \Omega_1$ and $H(x, a_0, Z_0), Z_0 \in \Sigma$ is a supporting hyperboloid of u_2 at x_0 , i.e. $Z_0 \in \mathcal{R}_{u_2}(x)$. Let us show that there is \bar{x} such that $Z_0 \in \mathcal{R}_{u_1}(\bar{x})$. Observe that by (2.11) the hyperboloids $H(x, a_0 + s, Z_0)$ stay above \mathcal{U} for all $s < 0$ such that $\bar{a}_0 > a_0 + s$, see (9.6) for the definition of \bar{a}_0 . Notice that if $a_0 + s > 0$ is very small then the corresponding hyperboloid $H(x, a_0 + s, Z_0)$ is very close to the asymptotic cone of the hyperboloid, with vertex very close to the fixed focus Z_0 .

Consequently, from the confocal expansion of hyperboloids (see subsection 8.4) we infer that there is $s_0 < 0$ such that $H(x, a_0 + s, Z_0)$ is a supporting hyperboloid of u_1 at an interior point $x_1 \in \Omega_1$. Thus $H(x, a_0 + s_0, Z_0)$ is a local supporting hyperboloid of u_1 . Since Γ_{u_1} is in the regularity domain \mathcal{D} , where (2.4)-(2.8) are fulfilled, we can apply Lemma 10.1 to conclude that $H(x, a_0 + s, Z_0)$ is also a global supporting hyperboloid of u_1 . Hence $Z_0 \in \mathcal{R}_{u_1}(x_1)$. Therefore

$$\mathcal{R}_{u_2}(\Omega_1) \subset \mathcal{R}_{u_1}(\Omega_1)$$

implying

$$\int_{\Omega_1} f_1 dx < \int_{\Omega_1} f_2 dx = \int_{\mathcal{R}_{u_2}(\Omega_1)} g d\mathcal{H}^n \leq \int_{\mathcal{R}_{u_1}(\Omega_1)} g d\mathcal{H}^n = \int_{\Omega_1} f_1 dx$$

which gives a contradiction. Thus $\Omega_1 = \emptyset$. □

13.1. Discrete Dirichlet problem. To outline our next two steps, we note that for the classical Monge-Ampère equation the standard way of proving the existence of globally smooth solutions to Dirichlet's problem with, say, $\varphi \in C^4(\overline{D})$ is to employ the continuity method combined with standard mollification argument, see [20]. Moreover, in this argument φ must be a subsolution. In order to tailor a similar proof for (13.3) we will mollify our weak A -solution, add $K(r^2 - |x - x_0|^2), K \gg 1$ and consider its restriction to $B_r(x_0) \subset \mathcal{U}$, a ball with sufficiently small radius. Such function turns out to be classical subsolution for some large K and small $r > 0$. Consequently, from continuity method one can obtain existence of a smooth solution to Dirichlet's problem in $B_r(x_0)$. Finally employing the known C^2 a priori estimates and comparison principle, Proposition 13.1, the proof of Theorem D will follow, see Section 14 for more details. Our approach most closely follows that proposed by Xu-Jia Wang [27].

Let $\{b_i\} \subset \partial D$ be a sequence of points on the boundary of D and $\{a_i\} \subset D$. Let $A_N = \{a_1, \dots, a_N\}$ and $B_N = \{b_1, \dots, b_N\} \subset \partial D$, for each fixed $N \in \mathbb{N}$. Furthermore, let $\nu_k(x)$ be atomic measures supported at $a_k, 1 \leq k \leq N$ and let

$$(13.4) \quad \mathcal{F}[v](x) = \nu_k(x) \frac{f(x)}{g \circ \mathcal{R}_v(x)}.$$

Proposition 13.2. *Let $\underline{u} \in \overline{\mathbb{W}}_H^0(D, \Sigma)$ be a polyhedral subsolution of (13.4), i.e. $\mathcal{F}[\underline{u}](x) \geq \nu_k(x) \frac{f(x)}{g \circ \mathcal{R}_{\underline{u}}(x)}$ at $a_k \in A_N$. Then there is a unique A -type weak solution to (13.4) verifying the boundary condition $u = \underline{u}$ on B_N .*

Proof. We want to construct a sequence of subsolutions $\{u_m\}_{m=0}^\infty$ converging to the solution of discrete problem. Set $u_0 = \underline{u}$ and define u_1 such that $u_1 \leq u_0$ in A_N , $u_1(b_i) = u_0(b_i), b_i \in B_N$ and $\alpha_{u_1, g}(a_i) \leq \alpha_{u_0, g}(a_i)$ for $a_i \in A_N$. It is convenient to introduce the class of hyperboloids

$$\Phi_{0, \delta}(a_1) = \left\{ P \in \mathbb{H}_0^+(D, \Sigma) : \begin{array}{l} H(a_i) \geq u_0(a_i), i \neq 1, \\ H(a_1) \geq u_0(a_1) - \delta, \\ H(b_j) \geq u_0(b_j), 1 \leq j \leq N \end{array} \right\}$$

for $\delta > 0$ and let

$$T_1^\delta u_0 = \inf_{H \in \Phi_{0, \delta}(a_1)} H.$$

Let $\delta_1 > 0$ be the largest δ for which $T_1^{\delta_1} u_0$ is a subsolution to (13.4) on A_N . Consequently, by setting $u_{0,1} = T_1^{\delta_1}$ we can proceed by induction and define the k -th class

$$\Phi_{0, \delta}(a_k) = \left\{ H \in \mathbb{H}_0^+(D, \Sigma) : \begin{array}{l} H(a_i) \geq u_{0, k-1}(a_i), i \neq k, \\ H(a_k) \geq u_{0, k-1}(a_k) - \delta, \\ H(b_j) \geq u_{0, k-1}(b_j), 1 \leq j \leq N \end{array} \right\}$$

and take $T_k^\delta u_0 = \inf_{H \in \Phi_{0, \delta}(a_k)} H$. Therefore, one can successively construct the functions $u_{0, k} = T_k^{\delta_k} u_{0, k-1}$ where $\delta_k > 0$ is the largest number for which $T_k^{\delta_k} u_{0, k-1}$ is a subsolution to (13.4) in A_N . Taking the second subsolution in the approximating sequence to be $u_2(x) \stackrel{\text{def}}{=} T_N^{\delta_N} u_{0, N-1}$ we get, by construction, that $\alpha_{u_0, g}(a_i) \leq \alpha_{u_1, g}(a_i)$, since we have the inclusions $\Phi_{l, \delta}(a_k) \subset \Phi_{l+1, \delta}(a_k)$ at a_k as we proceed. Therefore we have a sequence of solutions u_m to the Dirichlet problem in A_N such that

$$\begin{aligned} \alpha_{u_m, g}(a_i) &\leq \alpha_{u_{m-1}, g}(a_i), \\ u_m(a_i) &\leq u_{m-1}(a_i), \\ u_m(b_i) &= u_{m-1}(b_i). \end{aligned}$$

The first two inequalities are obvious. As for the boundary condition we note that $u_0(b_i) \leq u_1(b_i)$ by construction. If $u_0(b_i) < u_1(b_i)$ then by taking $\min[H_i(x), u_1(x)]$, where $H_i(x) \in \mathbb{H}_0^+(D, \Sigma)$ is a supporting hyperboloid of u_0 at b_i we see that $\min[H_i(x), u_1(x)]$ belongs to the corresponding Φ class. Thus $u_0(b_i) = u_1(b_i)$.

From Lemma 8.2 we conclude that $u \in \overline{\mathbb{W}}_H(D, \Sigma)$ and in view of Lemma 11.3 $\alpha_{u_m, g} \rightharpoonup \alpha_{u, g}$ weakly. Thus $u = \lim_{m \rightarrow \infty} u_m$ is a solution to the discrete problem in A_N with $u(b_i) = \underline{u}(b_i), b_i \in B_N$. \square

13.2. General case. Perron's method, used in the proof of above proposition, can be strengthened in order to establish the solvability of the general Dirichlet problem. To do so we take $\{a_i\}_{i=1}^\infty \subset D$ and $\{b_i\}_{i=1}^\infty \subset \partial D$ to be dense subsets and $A_N = \{a_1, \dots, a_N\} \subset D, B_N = \{b_1, \dots, b_N\} \subset \partial D$.

Proposition 13.3. *Let $\underline{u} \in \mathcal{AS}^+(D, \Sigma)$. Then there exists a unique weak solution u to the Dirichlet problem*

$$(13.5) \quad \begin{cases} \mathcal{F}[u] = \frac{f(x)}{g \circ \mathcal{R}_u(x)} & \text{in } D, \\ u(x) = \underline{u}(x) & \text{on } \partial D. \end{cases}$$

Proof. For $\delta > 0$ we denote $D_\delta = \{x \in D : \text{dist}(x, \partial D) > \delta\}$ and take $\eta(x)$ to be a smooth function such that $0 \leq \eta(x) \leq 1$, $\eta \equiv 1$ in $D_{2\delta}$ and $\eta \equiv 0$ in $D \setminus D_\delta$. Consider the equation

$$(13.6) \quad \mathcal{F}[v](x) = \nu_k(x) J(v(x)) \eta_\delta(x) \frac{f(x) - \delta}{g \circ \mathcal{R}_v(x)}$$

where $\nu_k(x)$ is a positive measure supported at $a_k \in A_N$ and

$$(13.7) \quad J(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq \sup_D \underline{u}, \\ \frac{2 \sup_D \underline{u} - t}{\sup_D \underline{u}} & \text{if } \sup_D \underline{u} \leq t \leq 2 \sup_D \underline{u}, \\ 0 & \text{if } t > 2 \sup_D \underline{u}. \end{cases}$$

Consider the class

$$(13.8) \quad \mathbb{W}_{N, \underline{u}}^+ = \left\{ v \in \overline{\mathbb{W}}_H^0(D, \Sigma) : \mathcal{F}[v] \geq \nu_k J(v) \eta_\delta \frac{f - \delta}{g \circ \mathcal{R}_u} \text{ and } v \geq \underline{u} \text{ on } B_N \right\}.$$

Clearly $\mathbb{W}_{N, \underline{u}}^+$ is not empty since $H(\cdot, a, Z)$ is in this class if $a > 0$ is sufficiently small. We claim that if $v_{N, \delta} = \inf_{\mathbb{W}_{N, \underline{u}}^+} v$ then $v_{N, \delta}$ solves (13.1) in the sense of Definition 13.1 and $v_{N, \delta}(b_i) = \underline{u}(b_i)$, $b_i \in B_N$.

It is easy to see that $\alpha_{v_{N, \delta}, g}(a_k) = v_k(a_k) J(v_{N, \delta}) \eta_\delta(a_k) (f(a_k) - \delta)$. Indeed, if $v_{N, \delta}$ is a strict subsolution at a_i , i.e. for some a_i we have $\alpha_{v_{N, \delta}, g}(a_i) > v_k(a_i) J(v_{N, \delta}) \eta_\delta(a_i) (f(a_i) - \delta)$, then we can push $\Gamma_{v_{N, \delta}}$ downward by some $\delta > 0$, decreasing the α measure at a_i , which, however, will be in contradiction with the definition of $v_{N, \delta}$. Thus $v_{N, \delta}$ is a solution of the equation (13.6).

Next, we check the boundary condition. Choose $H_i \in \mathbb{H}_0^+(\mathcal{U}, \Sigma)$ such that $H_i > v_\delta$ in \mathcal{U}_δ and passes through $(b_i, \underline{u}(b_i))$. Such H_i exists because by construction $v_{N, \delta}(a_i) \leq \underline{u}(a_i)$ and $\delta > 0$.

For $\tilde{H}_i = \min[H_i, v_{N, \delta}]$, by construction, we see that $\mathcal{F}[\tilde{H}_i] \geq \nu_k J(\tilde{H}_i) \eta_\delta \frac{f - \delta}{g \circ \mathcal{R}_{\tilde{H}_i}}$ at a_i . Thus $\tilde{H}_i \in \mathbb{W}_{N, \underline{u}}^+$. Hence

$$v_{N, \delta}(b_i) = \inf_{H \in \mathbb{W}_{N, \underline{u}}^+} H(b_i) \leq \tilde{H}_i(b_i) = \underline{u}(b_i).$$

Now the desired solution can be obtained via a standard compactness argument that utilizes the estimates of Lemma 8.1 and Lemma 11.3. More precisely, for fixed $\delta > 0$ we send $N \rightarrow \infty$ and obtain a function v_δ that solves the equation $\mathcal{F}[v_\delta] = J(v_\delta) \eta_\delta \frac{f - \delta}{g \circ \mathcal{R}_{v_\delta}}$. To show that $v_\delta = \underline{u}$ on ∂D we take $x_0 \in \partial D$ and again use the comparison with $\min[H_0, v_\delta]$ for a suitable $H_0 \in \mathbb{H}_0^+(\mathcal{U}, \Sigma)$ such that $H_0(x_0) = \underline{u}(x_0)$. Thus, from Proposition 13.1 we conclude that $v_\delta \leq \underline{u}$ in D . Finally sending $\delta \downarrow 0$ and employing the estimate of Lemmas 8.1 and 11.3 we arrive at desired result. \square

14. PROOF OF THEOREM D

To fix the ideas we assume that $x_0 = 0 \in \mathcal{U}$ and $B_r = B_r(0) \subset \mathcal{U}$. Let $u_{s, \delta}^\pm$ be the solutions to

$$(14.1) \quad \begin{cases} \mathcal{F}[u_{s, \delta}^\pm] = \frac{f \pm \delta}{h g \circ \mathcal{Z}_{u_{s, \delta}^\pm}} & \text{in } B_r \\ u_{s, \delta}^\pm = \tilde{u}_s & \text{on } \partial B_r \end{cases}$$

where $\tilde{u}_s = u_s + K(r^2 - |x|^2)$, $K > 0$ and u_s is a mollification of the weak solution u . By Lemma 10.2 \tilde{u}_s is a subsolution (for appropriate choice of constants K and r) and hence by Proposition 13.3 the weak solution to

Dirichlet's problem exists. Note that for the Dirichlet problem we have to consider the modified receiver $\tilde{\Sigma}$ to guarantee that \tilde{u}_s is admissible, see Lemma 10.2. In order to show the existence of smooth solutions we apply the continuity method: Let $\underline{w} \in \mathcal{AS}^+(B_r, \tilde{\Sigma}) \cap C^\infty(B_r)$ and for $t \in [0, 1]$ consider the solutions to the Dirichlet problem

$$(14.2) \quad \begin{cases} \mathcal{F}[w^t] = t \frac{f}{h \circ Z_w} + (1-t)\mathcal{F}[\underline{w}] & \text{in } B_r, \\ w^t = \underline{w} & \text{on } \partial B_r, \end{cases}$$

where h is given by (6.4). Using the implicit function theorem, see [23] Theorem 5.1, we find that the set of t 's for which (14.2) is solvable is open.

Once $C^{1,1}$ global a priori estimates were established in $\overline{B_r}$ then one can deduce that the set of such t 's is also closed. Recall that if $\partial\Omega \in C^3$, $u \in C^4(\Omega) \cap C^3(\overline{\Omega})$ and $\underline{u} \in C^4$ then from global $C^{1,1}$ estimates and the elliptic regularity theory we obtain that $w \in C^{2,\alpha}(\overline{\Omega})$. Therefore the existence of smooth solutions $u_{s,\delta}^\pm$ will follow once we establish the global $C^{1,1}$ estimate for w . The latter follows from [9] and Theorem B.

Summarizing, we have that $u_{s,\delta}^\pm$ remain locally uniformly smooth in B_r . Letting $s \rightarrow 0$ and applying the comparison principle (see Proposition 13.1) we have that $u_{0,\delta}^- \leq u \leq u_{0,\delta}^+$ and $u_{0,\delta}^\pm = u$ on ∂B_r . It follows from the a priori estimates in [9] and [18] that $u_{0,\delta}^\pm$ are locally uniformly C^2 in B_r for any small $\delta > 0$. After sending $\delta \rightarrow 0$ we will conclude the proof of Theorem D.

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MAXWELL INSTITUTE FOR MATHEMATICAL SCIENCES AND SCHOOL OF MATHEMATICS, UNIVERSITY OF EDINBURGH, JAMES CLERK MAXWELL BUILDING, KING'S BUILDINGS, MAYFIELD ROAD, EH9 3JZ, EDINBURGH, SCOTLAND
E-mail address: aram.karakhanyan@ed.ac.uk